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CONTENTS

New foundation of Euclidean geometry. By KARL MENGER, . . 721
Note on the Green function of a star-shaped three dimensional region. By J. J. GERGEN, ... 746
A junction property of locally connected sets. By G. T. WHYBURN, . 753
On the analytical extension of functions defined by factorial series. By HOWARD K. HUGHES, ... 757
On separation, comparison and oscillation theorems for self-adjoint systems of linear second order differential equations. By G. A. BLISS and I. J. SCHOENBERG, ... 781
On identities arising from solutions of q-difference equations and some interpretations in number theory. By G. W. STARCHER, . 801
Summation factors which are powers of a complex variable. By WALTER H. DURFEE, ... 817
A new calculus of numerical functions. By D. H. LEHMER, . . 843
On certain identities in theta functions. By RUSSELL SMITH PARK, . 855
On the zeros of a polynomial and of its derivative. By HUBERT E. BRAY, 864
On the numerical calculation of the current in an antenna. By F. H. MURRAY, ... 873
On the groups generated by two operators of orders two and three whose product is of order eight. By H. R. BRAHANA, . . 891
On a new normal form of the general cubic surface. By ARNOLD EMCH, 902
On plane Cremona triadic characteristics. By CHARLES C. TORRANCE, 911
A determination of the domains of integrity of the complete rational matric algebra of order 4. By E. J. FINAN, . . . 920
Pencils of hypersurfaces. By TEMPLE RICE HOLLCROFT, . . . 929

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E. A. Moore June 1930

NEW FOUNDATION OF EUCLIDEAN GEOMETRY.

By KARL MENGER.

My second paper on metrical geometry * contains a characterisation of the n-dimensional euclidean space among general semi-metrical spaces in terms of relations between the distances of its points. In courses on metrical geometry at American universities I have considerably shortened and revised my original proofs and generalized the formulations by introducing the concept of congruence order. The following paper contains these new proofs. In the first part we prove that every semi-metrical space, each $n+3$ points of which are congruent with $n+3$ points of the n-dimensional euclidean space, is congruent with a subset of the n-dimensional euclidean space. This is expressed by saying that the n-dimensional euclidean space has the congruence order $n+3$. In the second part we prove that each semi-metrical space containing more than $n+3$ points each $n+2$ points of which are congruent with $n+2$ points of the n-dimensional euclidean space, is congruent with a subset of the n-dimensional euclidean space. This fact is expressed by saying that the R_n has the quasi-congruence order $n+2$. It is proved by a systematic study of those sets which contain exactly $n+3$ points and are not congruent with $n+3$ points of the n-dimensional euclidean space whereas each $n+2$ of them are congruent with $n+2$ points of the n-dimensional euclidean space. These sets are called pseudo-euclidean sets. By means of these results the problem is reduced to the question: under what conditions are $n+2$ points congruent with $n+2$ points of the R_n and by what distance relations are the pseudo-euclidean $(n+3)$-tuples characterized. These purely algebraic problems are solved in the third part.†

I. CONGRUENCE SYSTEMS AND CONGRUENCE ORDER OF THE R_n. By a *congruence system* is meant a system $\mathfrak{S}$ of sets and a relation $\simeq$ (called the congruence relation) that satisfy the following five postulates:

Postulate 1. If p, q are two points of a set M of $\mathfrak{S}$, and p', q' are two points of a set M' of $\mathfrak{S}$ (M' not necessarily distinct from M) then, either $p, q \simeq p', q'$ or $p, q \not\simeq p', q'$, where the relation $\not\simeq$ is the negative of the relation of congruence.

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* Cp. Menger, *Mathematische Annalen*, Vol. 100, p. 113.

† I wish to express my thanks to Dr. Leonard M. Blumenthal for his help in the editing of this paper and in particular for remarks concerning the first part.

Postulate 2. For each two points p, q of a set of $\mathfrak{S}$, $p, q \simeq q, p$.

Postulate 3. For each point p of a set of $\mathfrak{S}$ and each two points q and r of a set of $\mathfrak{S}$, $p, p \simeq q, r$ if and only if $q = r$.

Postulate 4. If for two pairs of points $p, q;\ p', q'$ of two sets of $\mathfrak{S}$, $p, q \simeq p', q'$ then $p', q' \simeq p, q$.

Postulate 5. If for three pairs of points $p, q;\ p', q';\ p'', q''$ of three sets of $\mathfrak{S}$, $p, q \simeq p', q'$ and $p, q \simeq p'', q''$, then $p', q' \simeq p'', q''$.

A set S is called a *semi-metrical space* provided that to each two elements p, q of S there corresponds a not negative real number, called the distance between the points p and q, such that, if we denote this number by pq, we have $pq = qp$, and $pq = 0$ if and only if the points p and q are identical.

In particular, for each integer n the R_n (i. e., the n-dimensional Euclidean space) is a semi-metrical space. The system of all semi-metrical spaces is a congruence system if the relation $p, q \simeq p', q'$ is valid if and only if for the distances $pq, p'q'$ the equality $pq = p'q'$ subsists.

A mapping of the set S of the congruence system $\mathfrak{S}$ upon the set S' of $\mathfrak{S}$ is called a *congruent mapping* if to each pair of points of S there corresponds a congruent pair of points of S'. (A congruent mapping is, *eo ipso*, one-to-one). Two sets of a congruence system are called *congruent* if there exists a congruent mapping of one upon the other.

A set M of a congruence system $\mathfrak{S}$ has the *congruence order* n provided that *each set of* $\mathfrak{S}$, *each* n *points of which are congruent to* n *points of* M, *is congruent to a subset of* M. Evidently, each set of the congruence order n has also congruence order m, if m is an integer greater than n. For example, the R_0 (i. e., the space consisting of a single point) has the congruence order 2. In general, each set R of a congruence system $\mathfrak{S}$, consisting of n points has the congruence order $n + 1$, for if S is a set of $\mathfrak{S}$, each $n + 1$ points of which are congruent to $n + 1$ points of R, then S cannot contain more than n points.

Let S and R be two sets of a congruence system. If R has the congruence order n, the set S is said to be *super-ordered* with respect to R if the following conditions are satisfied:

(a) S contains a subset congruent to R.

(b) Each set of n points of S, every $(n-1)$ of which are congruent to $(n-1)$ points of R, is congruent to n points of R.

(c) If $p_1, p_2, \cdots, p_{n-1}$ are $(n-1)$ points of S, and $\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_{n-1}$

are $(n-1)$ points of S congruent to the first set of $(n-1)$ points, then to each point s of S there exists at least one point $\bar{s}$ of S such that

$$p_1, p_2, \cdots, p_{n-1}, s \simeq \bar{p}_1, \bar{p}_2, \cdots, \bar{p}_{n-1}, \bar{s}.$$

(*d*) If $p_1, p_2, \cdots, p_{n-1}$ and $\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_{n-1}$ are congruent sets of points of S, the first set not congruent to $(n-1)$ points of R, and if s, t and $\bar{s}, \bar{t}$ are two pairs of points of S such that

$$s, p_1, p_2, \cdots, p_{n-1} \simeq \bar{s}, \bar{p}_1, \bar{p}_2, \cdots, \bar{p}_{n-1}$$
$$t, p_1, p_2, \cdots, p_{n-1} \simeq \bar{t}, \bar{p}_1, \bar{p}_2, \cdots, \bar{p}_{n-1}$$

then $s, t \simeq \bar{s}, \bar{t}$.

It is to be observed that applying condition (d) to the case in which the points s and t are identical we see that the point s of condition (*c*) is uniquely determined. This remark we shall denote by (*c'*). It is of great use in what follows.

Remark 1. If R is the $(n-3)$-dimensional Euclidean space, and S is the $(n-2)$-dimensional Euclidean space, then R and S satisfy the conditions (*a*) — (*d*).*

(*a*) Each $(n-3)$-dimensional plane of the R_{n-2} is congruent to R_{n-3}.

(*b*) Each set of n points of the R_{n-2}, every $(n-1)$ of which are congruent to $(n-1)$ points of the R_{n-3} is congruent to n points of R_{n-3}.

(*c*) In applying a congruent self-transformation to the R_{n-2} which transforms $p_1, p_2, \cdots, p_{n-1}$ into $\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_{n-1}$ respectively, the point s is transformed into a point $\bar{s}$ satisfying the condition.

(*d*) If $p_1, p_2, \cdots, p_{n-1}$ are $(n-1)$ points of the R_{n-2} not lying in an $(n-3)$-dimensional plane, then if $\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_{n-1}$ are congruent to $p_1, p_2, \cdots, p_{n-1}$ there exists only one congruent self-transformation of the R_{n-2} transforming $p_1, p_2, \cdots, p_{n-1}$ into $\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_{n-1}$. Applying this self-transformation to the R_{n-2} the points s and t are necessarily transformed into $\bar{s}$ and $\bar{t}$ respectively; for the points into which s and t are transformed have the same distances from $\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_{n-1}$ as $\bar{s}$ and $\bar{t}$ have, and the R_{n-2} contains at most one point with $(n-1)$ given distances from $(n-1)$ points not lying in an $(n-3)$-dimensional plane.

Let us denote by S_n the n-dimensional sphere; for example, S_0 is a pair of points, S_1 is the circumference of a circle, S_2 is the surface of the sphere in three-space, etc. The section of an S_n by an m-dimensional plane $(m \leq n)$

* As soon as we shall have proved that the R_{n-3} has the congruence order n, it will follow from this remark that the R_{n-2} is super-ordered with respect to the R_{n-3}.

through the center of the $(n+1)$-dimensional sphere whose surface is S_n is called an $(m-1)$-dimensional great sphere of S_n.

Remark 2. If R is the S_{n-3}, and S the S_{n-2}, then the conditions (a)—(d) are satisfied.

(a) Each $(n-3)$-dimensional great sphere of S_{n-2} is congruent to S_{n-3}.

(b) Each set of n points of S_{n-2}, every $(n-1)$ of which are congruent to $(n-1)$ points of S_{n-3}, is congruent to n points of S_{n-3}.

(c) We may transform the sphere S_{n-2} congruently so that the points $p_1, p_2, \cdots, p_{n-1}$ are transformed into the congruent set $\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_{n-1}$. (If the two sets of points form two symmetric polygons, this transformation may be accomplished by means of a perspectivity with the center of the sphere as center of perspectivity). Then the point s is transformed into a point $\bar{s}$ satisfying the condition.

(d) If $p_1, p_2, \cdots, p_{n-1}$ are $(n-1)$ points of the S_{n-2} not lying in a $(n-3)$-dimensional great sphere, there exists only one congruent self-transformation of the S_{n-2} transforming the points $p_1, p_2, \cdots, p_{n-1}$ into the congruent set $\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_{n-1}$. The points s and t are transformed by this transformation into points $\bar{s}$ and $\bar{t}$ respectively, with $s, t \approx \bar{s}, \bar{t}$.

We now prove the

THEOREM. *If R has the congruence order n and S is super-ordered with respect to R, then S has the congruence order $n+1$.*

Given a set S', each $n+1$ points of which are congruent to $n+1$ points of S, we must show that S' is congruent to a subset of S. Now, since each $n+1$ points of S' are congruent to $n+1$ points of S, *a fortiori*, each n points of S' are congruent to n points of S.

Suppose, first, that each n points of S' are congruent with n points of R. Then since R has the congruence order n, S' is congruent with a subset of R. But S contains a subset congruent to R (condition (a)). Hence, S' is congruent to a subset of S, and the theorem is proved for this case.

Suppose, now, that there exist n points of S' that are not congruent to n points of R. There exist, however, n points of S congruent to these n points of S'. Among these n points of S there are $(n-1)$ points that are not congruent to $(n-1)$ points of R (since otherwise, by condition (b), the n points of S would be congruent to n points of R, and the n points of S' would be congruent to n points of R).

We may label the n points of S congruent to these n points of S' so that

(*) $\qquad s_1, s_2, \cdots, s_{n-1}$

are not congruent to $(n-1)$ points of R. Let $s'_1, s'_2, \cdots, s'_{n-1}$ be the $(n-1)$ points of S' congruent to the points (*) of S.

We construct, now, a mapping of S' upon a subset of S. Let s' be an arbitrarily chosen point of S'. Consider the set $s'_1, s'_2, \cdots, s'_{n-1}, s'$. These points are congruent to some n points of S, say, $\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_{n-1}, \bar{s}$. Now the set $s'_1, s'_2, \cdots, s'_{n-1}$ is congruent also to the set (*). Hence

$$s_1, s_2, \cdots, s_{n-1} \simeq \bar{s}_1, \bar{s}_2, \cdots, \bar{s}_{n-1}$$

where the first $(n-1)$-tuple is not congruent to $(n-1)$ points of R.

Now, according to the remark (c'), there is exactly one point s of S such that

$$s_1, s_2, \cdots, s_{n-1}, s \simeq \bar{s}_1, \bar{s}_2, \cdots, \bar{s}_{n-1}, \bar{s}$$

and hence

$$s_1, s_2, \cdots, s_{n-1}, s \simeq s'_1, s'_2, \cdots, s'_{n-1}, s'.$$

The point s determined in this way we take as the image in S of the point s' for each point s' of S'.

We propose that this mapping of S' on a subset of S is a congruent one. To show this, let s' and t' be two points of S', and let s and t be their images in S. We prove that $s, t \simeq s', t'$. Since each $n+1$ points of S are congruent to $n+1$ points of S', there exist some $n+1$ points of S, say, $\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_{n-1}, \bar{s}, \bar{t}$ congruent with the $n+1$ points, $s'_1, s'_2, \cdots, s'_{n-1}, s', t'$ of S'. Now, $\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_{n-1} \simeq s_1, s_2, \cdots, s_{n-1}$. Furthermore, since $\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_{n-1}\bar{s} \simeq s'_1, s'_2, \cdots, s'_{n-1}, s'$, there subsists the relation $s_1, s_2, \cdots, s_{n-1}, s \simeq \bar{s}_1, \bar{s}_2, \cdots, \bar{s}_{n-1}, \bar{s}$, where the first $(n-1)$-tuple is not congruent to $(n-1)$ points of R.

In entirely analogous fashion, we show that $s_1, s_2, \cdots, s_{n-1}, t \simeq \bar{s}_1, \bar{s}_2, \cdots, \bar{s}_{n-1}, \bar{t}$. Hence, we conclude from condition (d) that $s, t \simeq \bar{s}, \bar{t}$ and as $\bar{s}, \bar{t} \simeq s', t'$, it follows that $s, t \simeq s', t'$.

Thus we have shown that the mapping of S' upon a subset of S is a congruent one, and the theorem is proved.

We shall prove now:

FIRST FUNDAMENTAL THEOREM. *The R_n and the S_n have the congruence order $n+3$ but, except in the case of the R_0, not the congruence order $n+2$.*

As previously remarked, the R_0 has the congruence order 2 and, therefore, the congruence order 3. In order to prove that the R_n has the congruence order $n+3$ we show: If the R_{n-3} has the congruence order n, then the R_{n-2} has the congruence order $n+1$. But if we assume that the R_{n-3} has the congruence order n it follows from Remark 1 that the R_{n-2} is superordered

with respect to the R_{n-3}. Hence according to the Theorem we have proved, the R_{n-2} has the congruence order $n+1$. In the same way utilizing Remark 2 we conclude that the S_{n-2} has the congruence order $n+1$ if the S_{n-3} has the congruence order n. As the S_0 which consists of two points has the congruence order 3 it follows that the S_n has the congruence order $n+3$.

Let us now prove that the S_n has not a congruence order less than $n+3$. It is easy to prove that S_n contains $n+2$ points each two of which have the same distance, let us say, d, whereas S_n does not contain $n+3$ points, each two of which have the same distance. If, now, T is a set consisting of k points each two of which have the same distance d, then each $n+2$ points of T are congruent with $n+2$ points of S_n, whereas T, if $k \geq n+3$ is not congruent to a subset of S_n. Hence S_n has not the congruence order $n+2$.

Let us denote by $p_1, p_2, \cdots, p_{n+1}$ $n+1$ points of the R_n, each two of which have the same distance, say, d. We denote by d' the distance of the point p_{n+2} which is the center of the $(n-1)$-dimensional sphere that circumscribes the points $p_1, p_2, \cdots, p_{n+1}$, and by d'' the distance of p_{n+2} from the plane through $p_1, p_2, \cdots, p_n$. Then, let P be a metrical space consisting of $n+3$ points, $\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_{n+1}, \bar{p}_{n+2}, \bar{p}_{n+3}$ such that

$$\bar{p}_i\bar{p}_j = d, \qquad (i, j \leq n+1)$$

$$\bar{p}_{n+2}\bar{p}_i = \bar{p}_{n+3}\bar{p}_i = d' \qquad (i = 1, 2, \cdots, n+1).$$

$$\bar{p}_{n+2}\bar{p}_{n+3} = 2d''.$$

It is easy to prove that each $n+2$ points of P are congruent with $n+2$ points of the R_n, whereas P is not congruent with a subset of the R_n. Hence the R_n has not the congruence order $n+2$.

2. Pseudo-Euclidean Sets and Quasi-congruence Order. We have seen that there exists a set P which is not congruent to a subset of R_n whereas each $n+2$ points of P are congruent to $n+2$ points of the R_n. This set consisted of exactly $n+3$ points. We shall prove now in general:

Each set containing more than $n+3$ points, each $n+2$ of which are congruent to $n+2$ points of the R_n, is congruent to a subset of the R_n. We could say that a set S of a congruence system has *the quasi-congruence order k if each set containing more than $k+1$ points, each k points of which are congruent to k points of S, is congruent to a subset of S.* Then evidently, each set of quasi-congruence order k has the congruence order $k+1$. Conversely, however, the S_n is a space of congruence order $n+3$ which has not the quasi-congruence order $n+2$, for we have seen that for each integer k there exists a set T containing k points which is not congruent with a

subset of S_n whereas each $n+2$ points of T are congruent with $n+2$ points of S_n. Utilizing the concept of quasi-congruence order, we shall prove the

SECOND FUNDAMENTAL THEOREM. *The R_n but not the S_n has the quasi-congruence order $n+2$.*

The R_0 has the congruence order 2 and hence, a fortiori, the quasi-congruence order 2. We shall prove the theorem by complete induction, i. e. under the assumption of its truth for the R_{n-1}.

The following terminology will be useful: We call $n+1$ points of the R_n *independent* if and only if they do not lie in a $(n-1)$-dimensional plane. We say in this case also that each of these $n+1$ points is independent of the n others. More generally we shall call a set of a congruence system which is congruent with $n+1$ independent points of the R_n an independent $(n+1)$-tuple. A set of a congruence system that consists of $n+1$ points which are congruent with $n+1$ points of a R_{n-1} will be called a *dependent* $(n+1)$-tuple. We say, furthermore, that n points $p_1, \cdots, p_n$ of the R_n are independent if and only if there exists a point p_{n+1} of the R_n such that the points $p_1, \cdots, p_n, p_{n+1}$ are independent. In this notation the properties (a) — (d) of the R_n which guaranteed its congruence order $n+3$ can be formulated in the following way:

(a) The $(n-1)$-dimensional planes of the R_n are congruent with the R_{n-1}.

(b) $n+2$ points of the R_n each $n+1$ of which are dependent lie in a $(n-1)$-dimensional plane.

(c) If in the R_n the two $(n+1)$-tuples $p_1, \cdots, p_{n+1}$ and $\bar{p}_1, \cdots, \bar{p}_{n+1}$ are congruent then there exists for each point p of the R_n at least one point $\bar{p}$ of the R_n such that $p_1, \cdots, p_{n+1}, p \simeq \bar{p}_1, \cdots, \bar{p}_{n+1}, \bar{p}$.

(c') If the points $p_1, \cdots, p_{n+1}$ are independent than the point $\bar{p}$ in condition (c) is uniquely determined.

(d) If the points $\bar{p}_1, \cdots, \bar{p}_{n+1}$ of the R_n are independent and $\bar{p}_{n+2}$, $\bar{p}_{n+3}$ and $p_1, \cdots, p_{n+1}$, p_{n+2}, p_{n+3} are points of the R_n such that

$$\bar{p}_1, \cdots, \bar{p}_{n+1}, \bar{p}_{n+2} \simeq p_1, \cdots, p_{n+1}, p_{n+2}$$
$$\bar{p}_1, \cdots, \bar{p}_{n+1}, \bar{p}_{n+2} \simeq p_1, \cdots, p_{n+1}, p_{n+3}$$

then

$$\bar{p}_{n+2}\bar{p}_{n+3} \simeq p_{n+2}p_{n+3}.$$

An immediate consequence of (c) is besides

(c'') If in the R_n the two n-tuples $p_1, \cdots, p_n$ and $\bar{p}_1, \cdots, \bar{p}_n$ are congruent then there exists for each two points p and q of the R_n at least one point $\bar{p}$ and at least one point $\bar{q}$ such that

$$p_1, \cdots, p_n, p, q \approx \bar{p}_1, \cdots, \bar{p}_n, \bar{p}, \bar{q}.$$

Besides these properties of the R_n we shall make use in what follows of some other properties viz:

(e) Two congruent $(n+1)$-tuples of the R_n are either both independent or both dependent.

(f) If the points $p_1, \cdots, p_n$ of the R_n are independent (i. e. do not lie in a $(n-2)$-dimensional plane) then there exists for each point p which is independent of them at least one point $p^0 \neq p$ such that

$$p_1, \cdots, p_n, p \approx p_1, \cdots, p_n, p^0.$$

(g) If $p_1, \cdots, p_n$ are independent points of the R_n and

$$p_1, \cdots, p_n, p \approx p_1, \cdots, p_n, p^0$$
$$p_1, \cdots, p_n, q \approx p_1, \cdots, p_n, q^0$$

where p^0 is either dependent on $p_1, \cdots, p_n$ or distinct from p, and where q^0 is either dependent on $p_1, \cdots, p_n$ or distinct from q, then

$$p_1, \cdots, p_n, p, q \approx p_1, \cdots, p_n, p^0, q^0.$$

If in condition (g) p and q are identical it follows that condition (f) can be sharpened to

(f') The point p^0 in condition (f) is uniquely determined.

(h) If in the R_n the points $p_1, \cdots, p_n$ are independent and p is dependent on them, then from

$$p_1, \cdots, p_n, p \approx p_1, \cdots, p_n, p^0$$

it follows that $p = p^0$.

(i) * If in the R_n the points $p_1, \cdots, p_{n+1}$ are independent and p is some other point and if we denote for $k = 1, 2, \cdots, n+1$ by p^k a point such that

$$p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p^k \approx p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p$$

and such that $p^k \neq p$ if p is independent of $p_1, \cdots, p_{n+1}$ then there exists at most one point c of the R_n such that

$$c, p^1 \approx c, p^2 \approx \cdots \approx c, p^{n+1}.$$

Proof: The $(n+1)$ independent points $p_1, p_2, \cdots, p_{n+1}$ in R_n determine a simplex. The $(n+1)$ points $p^1, p^2, \cdots, p^{n+1}$ obtained as the mirror images of an $(n+2)$ *nd* point, p, in the $(n+1)$ planes of the simplex lie, in general,

* Property (i) has been found and proved by Leonard M. Blumenthal.

in R_n, and not in a space of lower dimensions. There is then but one point, c, of the R_n satisfying the relation

$$(*) \qquad c, p^1 \approx c, p^2 \approx \cdots \approx c, p^{n+1}.$$

It may happen, however, that the points $p^1, p^2, \cdots, p^{n+1}$ lie in an $(n-1)$-dimensional plane (the generalization of the Simpson Line that occurs in the R_2). If the points do not lie in an $(n-2)$-dimensional plane, then there is no point, c, satisfying (*), in the $(n-1)$-dimensional plane bearing the points; for if such a point did exist, it is clear that the planes in which p is reflected would all pass through a point (the center of the n-dimensional sphere determined by the points $p^1, p^2, \cdots, p^{n+1}$ and the point p). It follows in this case that no point in R_n has the property (*) since the orthogonal projection of such a point upon the plane containing $p^1, p^2, \cdots, p^{n+1}$ would be a point in this plane satisfying (*).

If now, we let $k \leq n-1$ be the smallest dimension of the plane containing $p^1, p^2, \cdots, p^{n+1}$, then for $k < n-1$, the mirror planes are all orthogonal to this plane and hence do not form a simplex.†

(j) If in the R_n the points $p_1, \cdots, p_n, p_{n+1}$ are independent and p is a point such that for each integer $k = 1, 2, \cdots n$ the points $p_1, \cdots, p_{k-1}, p_{k+1} \cdots, p_n, p_{n+1}, p$ are dependent, then $p = p_{n+1}$.

(k) If q, r, s are three points of the R_n such that

$$q, r \approx r, s \approx s, q$$

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† An analytic proof may be given as follows: Let $(x_1{}^i, x_2{}^i, \cdots, x^i{}_{n-1})$ be the coördinates of the points p^i, $(i = 1, 2, \cdots, n+1)$, assumed to lie in an at most $(n-1)$-dimensional plane whose equation we may write as $X_n = 0$. We suppose, further, that the points lie on an at most $(n-1)$-dimensional sphere with equation $\sum_{k=1}^{n-1} X_k{}^2 = a^2$.

We show that the mirror planes do not, then, form a simplex. If $(x_1, x_2, \cdots, x_n)$ are the coördinates of the point p, the equations of the planes are

$$(x_1 - x_1{}^i)[X_1 - \tfrac{1}{2}(x_1 + x_1{}^i)] + (x_2 - x_2{}^i)[X_2 - \tfrac{1}{2}(x_2 + x_2{}^i)] + \cdots + x_n[X_n - \tfrac{1}{2}x_n] = 0 \qquad (i = 1, 2, \cdots, n+1)$$

and since the points p^i, $(i = 1, 2, \cdots, n+1)$ lie on the sphere $\sum_{k=1}^{n-1} X_k{}^2 = a^2$ the equations may be written

$$(x_1 - x_1{}^i)X_1 + (x_2 - x_2{}^i)X_2 + \cdots + x_n X_n = \tfrac{1}{2}[\sum_{k=1}^{n} x_k{}^2 - a^2].$$

To prove that these planes do not form an n-dimensional simplex it is sufficient to note that the determinant of these equations is zero.

and if $p_1, \cdots, p_n$ are n points of the R_n such that

$$p_i, q \cong p_i, r \cong p_i, s \qquad (i = 1, 2, \cdots n)$$

then the n points $p_1, \cdots, p_n$ are dependent, i. e. lie in a $(n-2)$ dimensional plane.

To prove this property we remark that the n points p_i lie on the surfaces of three spheres. If they would not lie in a $(n-2)$-dimensional plane the centres of these three spheres would lie on a straight line. The centres q, r, s of the three spheres form, however, according to the hypothesis an equilateral triangle and hence do not lie on a straight line.

We shall call a set S of a congruence system *equilateral* if each two couples of points p, q and r, s of S are congruent. Then we see

(l) The R_n does not contain any equilateral $(n+2)$-tuple.

An important rôle in the metrical theory of the euclidean space is played by the concept of *pseudo-euclidean sets.* A set S of a congruence system consisting of $n+3$ points is called a pseudo-euclidean $(n+3)$-tuple *if S is not congruent with a subset of the R_n whereas each $n+2$ points of S are congruent with $n+2$ points of the R_n.* We prove first of all the following:

LEMMA 1. *If the pseudo-euclidean $(n+3)$-tuple P' contains $n+2$ points, $n+1$ points of which are independent* (i. e. congruent with $n+1$ points of the R_n but not with $n+1$ points of the R_{n-1}) *and such that the $n+2$ points are congruent with $n+2$ points of the pseudo-euclidean $(n+3)$-tuple Q' then P' and Q' are congruent.*

Let $p'_1, \cdots, p'_{n+3}$ be the $n+3$ points of P' and $q'_1, \cdots, q'_{n+3}$ the points of Q'. Let us suppose that $p'_1, \cdots, p'_{n+2} \cong q'_1, \cdots, q'_{n+2}$ and that $p'_1, \cdots, p'_{n+1}$ are independent. We have then to prove that $p'_i, p'_{n+3} \cong q'_i, q'_{n+3}$ $(i = 1, \cdots, n+2)$. We study first some properties of P'.

As the set P' is pseudo-euclidean there exist $n+2$ points $p_1, \cdots, p_{n+1}$, p_{n+2} of the R_n such that

$$(1) \qquad p'_1, \cdots, p'_{n+1}, p'_{n+2} \cong p_1, \cdots, p_{n+1}, p_{n+2}.$$

According to the hypothesis the points $p_1, \cdots, p'_{n+1}$ are independent, i. e. do not lie in an $(n-1)$-dimensional plane. As the points $p'_1, \cdots, p'_{n+1}$, p'_{n+3} are congruent with $n+2$ points of the R_n there exists according to property (c') of the R_n exactly one point p_{n+3} of the R_n such that

$$(2) \qquad p'_1, \cdots, p'_{n+1}, p'_{n+3} \cong p_1, \cdots, p_{n+1}, p_{n+3}.$$

If p'_{n+2}, p'_{n+3} would be congruent with p_{n+2}, p_{n+3} the $n+3$ points of P' would be congruent with the points $p_1, \cdots, p_{n+2}, p_{n+3}$. As P' is a pseudo-euclidean set this is impossible. Hence we have

$$(3) \qquad p'_{n+2}, p'_{n+3} \not\simeq p_{n+2}, p_{n+3}.$$

Let k be one of the integers $1, 2, \cdots, n, n+1$. There exist $n+2$ points $\bar{p}_1, \cdots, \bar{p}_{k-1}, \bar{p}_{k+1}, \cdots, \bar{p}_{n+1}, \bar{p}_{n+2}, \bar{p}_{n+3}$ of the R_n such that

$$(4_k) \qquad \bar{p}_1, \cdots, \bar{p}_{k-1}, \bar{p}_{k+1}, \cdots, \bar{p}_{n+1}, \bar{p}_{n+2}, \bar{p}_{n+3} \simeq p'_1, \cdots, p'_{k-1}, p'_{k+1}, \cdots, p'_{n+1}, p'_{n+2}, p'_{n+3}.$$

We have $\bar{p}_1, \cdots, \bar{p}_{n+1} \simeq p_1, \cdots, p_{n+1}$ as either of these $(n+1)$-tuples is congruent with $p'_1, \cdots, p'_{n+1}$, and these $(n+1)$-tuples are independent.

We state now the following preliminary proposition: If p is any point such that

$$\bar{p}_1, \cdots, \bar{p}_{k-1}, \bar{p}_{k+1}, \cdots, \bar{p}_{n+1}, \bar{p}_{n+2} \simeq p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p,$$

then p is independent of the points $p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}$.

In order to prove this proposition we assume that the point p satisfying the above mentioned condition is dependent on $p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}$ and we deduce a contradiction from this assumption. According to (c'') there exists at least one point x and at least one point y such that

$$\bar{p}_1, \cdots, \bar{p}_{k-1}, \bar{p}_{k+1}, \cdots, \bar{p}_{n+1}, \bar{p}_{n+2}, \bar{p}_{n+3} \simeq p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, x, y.$$

As p is dependent on $p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}$ it follows from property (h) of the R_n that $x = p$. As

$$\bar{p}_1, \cdots, \bar{p}_{k-1}, \bar{p}_{k+1}, \cdots, \bar{p}_{n+1}, \bar{p}_{n+2} \simeq p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p_{n+2}$$

it follows from property (h), furthermore, that $p = p_{n+2}$. We have thus

$$(5) \qquad \bar{p}_1, \cdots, \bar{p}_{k-1}, \bar{p}_{k+1}, \cdots, \bar{p}_{n+1}, \bar{p}_{n+2}, \bar{p}_{n+3} \simeq p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p_{n+2}, y.$$

On the other hand, we have

$$p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, y \simeq \bar{p}_1, \cdots, \bar{p}_{k-1}, \bar{p}_{k+1}, \cdots, \bar{p}_{n+1}, \bar{p}_{n+3}$$

and hence according to (4_k) and (2)

$$p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, y \simeq p'_1, \cdots, p'_{k-1}, p'_{k+1}, \cdots, p'_{n+1}, p'_{n+3}.$$

As p_{n+2} is dependent on the points $p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}$ it follows from property (g) of the R_n that

$$p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p_{n+2}, y \simeq p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p_{n+2}, p_{n+3}$$

thus

(6) $$p_{n+2}, y \simeq p_{n+2}, p_{n+3}.$$

On the other hand it follows from (5) that $p_{n+2}, y \simeq \bar{p}_{n+2}, \bar{p}_{n+3}$ and thus, in consequence of (4_k)

(7) $$p_{n+2}, y \simeq p'_{n+1}, p'_{n+3}.$$

(6) and (7) contradict (3). We have thus proved our preliminary proposition.

Exactly in the same fashion we can prove: If q is a point such that

$$\bar{p}_1, \cdots, \bar{p}_{k-1}, \bar{p}_{k+1}, \cdots, \bar{p}_{n+1}, \bar{p}_{n+3} \simeq p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, q$$

then q is independent of $p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}$.

In consequence of (1) and (2) we have

$$\bar{p}_1, \cdots, \bar{p}_{k-1}, \bar{p}_{k+1}, \cdots, \bar{p}_{n+1}, \bar{p}_{n+2} \simeq p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p_{n+2}$$
$$\bar{p}_1, \cdots, \bar{p}_{k-1}, \bar{p}_{k+1}, \cdots, \bar{p}_{n+1}, \bar{p}_{n+3} \simeq p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p_{n+3}$$

The points p_{n+2} and p_{n+3} are thus independent of $p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}$. According to (f') there exists exactly one point which may be called $p^k{}_{n+2}$ and exactly one point which may be called $p^k{}_{n+3}$ satisfying the conditions

$$p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p_{n+2} \simeq p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p^k{}_{n+2}, p^k{}_{n+2} \neq p_{n+2}$$
$$p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p_{n+3} \simeq p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p^k{}_{n+3}, p^k{}_{n+3} \neq p_{n+3}.$$

According to property (c'') there exists at least one point p and at least one point q such that

$$\bar{p}_1, \cdots, \bar{p}_{k-1}, \bar{p}_{k+1}, \cdots, \bar{p}_{n+1}, \bar{p}_{n+2}, \bar{p}_{n+3} \simeq p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p, q.$$

According to (g) the point p is identical with either p_{n+2} or $p^k{}_{n+2}$, and q is identical with either p_{n+3} or $p^k{}_{n+3}$. As

$$p_{n+2}, p_{n+3} \simeq p^k{}_{n+2}, p^k{}_{n+3} \not\simeq p'_{n+2}, p'_{n+3} \text{ and } \bar{p}_{n+2}, \bar{p}_{n+3} \simeq p_{n+2}, p_{n+3}$$

it follows that either $p = p_{n+2}$ and $q = p^k{}_{n+3}$ or $p = p^k{}_{n+2}$ and $q = p_{n+3}$. According to (g)

$$p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p_{n+2}, p^k{}_{n+3} \simeq p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p^k{}_{n+2}, p_{n+3}.$$

We have thus

$$p^k{}_{n+2}, p_{n+3} \simeq p_{n+2}, p^k{}_{n+3} \simeq p'_{n+2}, p'_{n+3}.$$

Hence it is proved for each k $(k = 1, 2, \cdots n+1)$ that

$$p'_{n+2}, p'_{n+3} \approx p^k_{n+2}, p_{n+3}.$$

We consider now the pseudo-euclidean $(n+3)$-tuple Q'. According to the hypothesis we have

$$q'_1, \cdots, q'_{n+2} \approx p'_1, \cdots, p'_{n+2} \approx p_1, \cdots, p_{n+2}.$$

Also the $n+2$ points $q'_1, \cdots, q'_{n+2}$ of Q' are congruent with $n+2$ points of the R_n and there exists exactly one point q_{n+3} of the R_n such that

$$q'_1, \cdots, q'_{n+1}, q'_{n+3} \approx p'_1, \cdots, p'_{n+1}, p'_{n+3}.$$

The same arguments which proved that

$$p'_{n+2}, p'_{n+3} \approx p^k_{n+2}, p_{n+3} \qquad (k = 1, 2, \cdots, n+1)$$

prove that also

$$q'_{n+2}, q'_{n+3} \approx p^k_{n+2}, q_{n+3}. \qquad (n = 1, 2, \cdots, n+1).$$

From property (i) of the R_n it follows that $q_{n+3} = p_{n+3}$. Therefore we have

$$q'_i, q'_{n+3} \approx p_i, p_{n+3} \approx p'_i, p'_{n+3} \qquad (i = 1, 2, \cdots, n+1)$$

$$q'_{n+2}, q'_{n+3} \approx p^k_{n+2}, p_{n+3} \approx p'_{n+2}, p'_{n+3}.$$

This is the proposition of the Lemma 1.

We shall prove now the following:

THEOREM I ON PSEUDO-EUCLIDEAN SETS. *Each $n+1$ points of a pseudo-euclidean $(n+3)$-tuple are independent,* i. e. congruent with $n+1$ points of the R_n but not congruent with $n+1$ points of the R_{n-1}.

We prove first of all: If P' is a pseudo-euclidean $(n+3)$-tuple then P' contains at least *one* independent $(n+1)$-tuple. If all $(n+1)$-tuples of points of P' would be dependent then P' would be congruent with a subset of the R_{n-1} as the R_{n-1} by our inductive hypothesis has the quasi-congruence order $n+1$. Then P' would be a fortiori congruent with a subset of the R_n and this is not the case.

We label the points of P' so that $p'_1, \cdots, p'_{n+1}$ are independent. Let $p_1, \cdots, p_{n+1}$ $n+1$ be congruent points of the R_n. They do not lie in a $(n-1)$-dimensional plane. We form the points p^k_{n+2} $(k = 1, \cdots, n+1)$ which were considered in the proof of Lemma 1, for which

$$p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p^k_{n+2}, p_{n+3} \approx p'_1, \cdots, p'_{k-1}, p'_{k+1}, \cdots, p'_{n+1}, p'_{n+2}, p'_{n+3}.$$

Let us now suppose that the points $p'_1, \cdots, p'_{k-1}, p'_{k+1}, \cdots, p'_{n+1}, p'_{n+2}$ are dependent. Then the points $p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p^k_{n+2}$ are dependent, also. As the points $p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}$ are independent it follows in consequence of property (h) from

$$p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p^k_{n+2} \approx p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p_{n+2}$$

that $p^k_{n+2} = p_{n+2}$. Then $p_{n+2}, p_{n+3} \approx p^k_{n+2}, p_{n+3} \approx p'_{n+2}, p'_{n+3}$ and this contradicts (3). The assumption has led to a contradiction and we have thus proved:

If the points $p'_1, \cdots, p'_{n+1}$ of the pseudo-euclidean $(n+3)$-tuple $p'_1, \cdots, p'_{n+3}$ are independent then each of the $(n+1)$-tuples $p'_1, \cdots, p'_{k-1}, p'_{k+1}, \cdots, p'_{n+1}, p'_{n+2}$ $(k = 1, 2, \cdots, n+1)$ is independent. For reasons of symmetry it follows that also each of the $(n+1)$-tuples $p'_1, \cdots, p'_{k-1}, p'_{k+1}, \cdots, p'_{n+1}, p'_{n+3}$ $(k = 1, 2, \cdots, n+1)$ is independent.

The independence of the $(n+1)$-tuple $p'_1, \cdots, p'_{i-1}, p'_{i+1}, \cdots, p'_{k-1}, p'_{k+1}, \cdots, p'_{n+1}, p'_{n+2}, p'_{n+3}$ follows from the independence of the points $p'_1, \cdots, p'_{i-1}, p'_{i+1}, \cdots, p'_{n+1}, p'_{n+2}$ in the same way as the independence of $p'_1, \cdots, p'_{k-1}, p'_{k+1}, \cdots, p'_{n+1}, p'_{n+2}$ follows from the independence of $p'_1, \cdots, p'_{n+1}$. Theorem I is thus proved.

Based on Theorem I we may formulate Lemma 1 as the following

THEOREM II ON PSEUDO-EUCLIDEAN SETS. *Two pseudo-euclidean $(n+3)$-tuples such that one contains a $(n+2)$-tuple congruent with a $(n+2)$-tuple of the other are congruent.*

We prove now the following:

Lemma 2. If a set consisting of $n+4$ distinct points each $n+2$ of which are congruent with $n+2$ points of the R_n, contains one pseudo-euclidean $(n+3)$-tuple it contains at least three pseudo-euclidean $(n+3)$-tuples $(n \geqq 1)$.

Let us call $p'_1, \cdots, p_{n+4}$ the $n+4$ distinct points of the set P' and assume that the points $p'_1, \cdots, p'_{n+3}$ form a pseudo-euclidean set. P' contains $n+4$ sets each consisting of $n+3$ points. If $n+2$ of these sets are pseudo-euclidean then, as $n \geqq 1$, the lemma is true. Otherwise there are at least two $(n+3)$-tuples each of which is congruent with $n+3$ points of the R_n. We may label the points of P' so that $p'_1, \cdots, p'_{n+1}, p'_{n+2}, p'_{n+4}$ and $p'_1, \cdots, p'_{n+1}, p'_{n+3}, p'_{n+4}$ are congruent to euclidean $(n+3)$-tuples. In this case we shall prove that there exist besides $p'_1, \cdots, p'_{n+3}$ at least two pseudo-euclidean $(n+3)$-tuples in P'.

First of all, there correspond to the pseudo-euclidean set $p'_1, \cdots, p'_{n+3}$, $n+3$ points $p_1, \cdots, p_{n+3}$ of the R_n such that

$$(1) \qquad p_1, \cdots, p_{n+1}, p_{n+2} \approx p'_1, \cdots, p'_{n+1}, p'_{n+2}$$

$$(2) \qquad p_1, \cdots, p_{n+1}, p_{n+3} \approx p'_1, \cdots, p'_{n+1}, p'_{n+3}$$

$$(3) \qquad p_{n+2}, p_{n+3} \not\approx p'_{n+2}, p'_{n+3}.$$

The points $p_1, \cdots, p_{n+1}$ are congruent to $n+1$ points of a pseudo-euclidean $(n+3)$-tuple. According to theorem I on pseudo-euclidean sets they do not lie in an $(n-1)$-dimensional plane. There exists therefore according to (c') exactly one point p_{n+4} in the R_n such that

$$(4) \qquad p_1, \cdots, p_{n+1}, p_{n+2}, p_{n+4} \approx p'_1, \cdots, p'_{n+1}, p'_{n+2}, p'_{n+4}$$

$$(5) \qquad p_1, \cdots, p_{n+1}, p_{n+3}, p_{n+4} \approx p'_1, \cdots, p'_{n+1}, p'_{n+3}, p'_{n+4}.$$

From (4) and (5) it follows that

$$p'_{n+4}, p'_i \approx p_{n+4}, p_i \qquad (i = 1, \cdots, n+3).$$

As the point p'_{n+4} is distinct from the $n+3$ points p'_i it follows thus that p_{n+4} is distinct from the $n+3$ points p_i $(i = 1, \cdots, n+3)$. According to property (j) of the R_n there exist therefore at least two integers i and j between 1 and $n+1$ such that neither $p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{n+1}, p_{n+4}$ nor $p_1, \cdots, p_{j-1}, p_{j+1}, \cdots, p_{n+1}, p_{n+4}$ lie in a $(n-1)$-dimensional plane. We propose now that

$$(6) \qquad p'_1, \cdots, p'_{i-1}, p'_{i+1}, \cdots, p'_{n+1}, p'_{n+2}, p'_{n+3}, p'_{n+4}$$

$$(7) \qquad p'_1, \cdots, p'_{j-1}, p'_{j+1}, \cdots, p'_{n+1}, p'_{n+2}, p'_{n+3}, p'_{n+4}$$

are the two desired pseudo-euclidean $(n+3)$-tuples.

Let us show this for the system (6). As each $(n+2)$-tuple of (6) is congruent with $n+2$ points of the R_n we have merely to show that (6) is not congruent with $n+3$ points of the R_n. If we map $p'_1, \cdots, p'_{i-1}, p'_{i+1}, \cdots, p'_{n+1}, p'_{n+4}$ on $p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{n+1}, p_{n+4}$ then, as these points do not lie in an $(n-1)$-dimensional plane the points p'_{n+2} and p'_{n+3} in consequence of (c') must be necessarily mapped on p_{n+2} and p_{n+3} respectively. Hence it is according to (3) impossible to map the $n+3$ considered points on $n+3$ points of the R_n. In the same way it can be shown that (7) is a pseudo-euclidean $(n+3)$-tuple. This completes the proof of lemma 2.

Lemma 3. If P' is a set consisting of $n+4$ distinct points each $n+2$ of which are congruent with $n+2$ points of the R_n and which contains at least three pseudo-euclidean $(n+3)$-tuples then each $(n+3)$-tuple of points of P' is pseudo-euclidean.

Let us denote by $p'_1, \cdots, p'_{n+4}$ the $n+4$ distinct points of P' and let us label them so that

(1) $\quad p'_1, \cdots, p'_{n+1}, p'_{n+2}, p'_{n+3}$

(2) $\quad p'_1, \cdots, p'_{n+1}, p'_{n+2}, p'_{n+4}$

(3) $\quad p'_1, \cdots, p'_{n+1}, p'_{n+3}, p'_{n+4}$

are pseudo-euclidean sets. As the $(n+3)$-tuples (1) and (2) both contain the $n+2$ points $p'_1, \cdots, p'_{n+2}$ it follows from theorem I on pseudo-euclidean sets that the $(n+3)$-tuples (1) and (2) are congruent. In the same way one sees that (1) and (3) are congruent. Hence we have

(4) $\quad p'_{n+2}, p'_{n+3} \simeq p'_{n+2}, p'_{n+4} \simeq p'_{n+3}, p'_{n+4}$

(5) $\quad p'_{n+2}, p'_i \simeq p'_{n+3}, p'_i \simeq p'_{n+4}, p'_i \qquad (i = 1, \cdots, n+1).$

We prove now that the $n+3$ points $p'_2, \cdots, p'_{n+4}$ form a pseudo-euclidean set. We assume that they are not pseudo-euclidean and deduce a contradiction from this assumption. As each $n+2$ of the points are congruent with $n+2$ points of the R_n it follows from the assumption that there exist $n+3$ points $p_2, \cdots, p_{n+4}$ of the R_n which are congruent with the points $p'_2, \cdots, p'_{n+4}$. In consequence of (4) and (5) the n points $p_2, \cdots, p_{n+1}$ according to property (k) of the R_n are dependent. This, however, is impossible as they are congruent with the n points of the pseudo-euclidean $(n+3)$-tuple (1) and, as no $n+1$ points of a pseudo-euclidean $(n+3)$-tuple are independent, also each n-tuple of points of (1) is independent.

Exactly in the same way we could deduce a contradiction from the assumption that any other $(n+3)$-tuple of P' is not pseudo-euclidean. Lemma 3 is thus proved.

We are now in the position to prove the third fundamental theorem. We have to prove that a set S containing more then $n+3$ points each $n+2$ of which are congruent with $n+2$ points of the R_n is congruent with a subset of the R_n. We shall assume that S is a set containing at least $n+4$ distinct points each $n+2$ of which are congruent with $n+2$ points of the R_n and deduce a contradiction from the hypothesis that S is not congruent with a subset of the R_n. As the R_n has the congruence order $n+3$ it follows from the hypothesis that S contains a $(n+3)$-tuple of points $p'_1, \cdots, p'_{n+3}$ which are not congruent with $(n+3)$ points of the R_n. Since each $n+2$ points of this $(n+3)$-tuple are congruent with $n+2$ points of the R_n it follows that they form a pseudo-euclidean $(n+3)$-tuple. As S contains at least $n+4$ distinct points there exists a point p'_{n+4} of S distinct from the points $p', \cdots, p'_{n+3}$. The set P' of the $n+4$ points $p'_1, \cdots, p'_{n+3}, p'_{n+4}$ contains at least one pseudo-euclidean $(n+3)$-tuple. According to lemma

2 the set P' contains at least three pseudo-euclidean $(n+3)$-tuples. According to lemma 3 each $(n+3)$-tuple of points of P' is pseudo-euclidean.

We prove now that the set P' is equilateral, i. e. each two pairs of points of P' are congruent. In order to prove that two pairs of points which have one point in common are congruent, e. g. that $p'_{n+2}, p'_{n+3} \simeq p'_{n+3}, p'_{n+4}$ we remark that the two pseudo-euclidean $(n+3)$-tuples

$$p'_1, \cdots, p'_{n+2}, p'_{n+3} \text{ and } p'_1, \cdots, p'_{n+2}, p'_{n+4}$$

have the $n+2$ points $p'_1, \cdots, p'_{n+2}$ in common and hence, according to theorem II on pseudo-euclidean sets, are congruent. This shows that in particular $p'_{n+2}, p'_{n+3} \simeq p'_{n+3}, p'_{n+4}$. In the same fashion it can be proved e. g. that $p'_{n+1}, p'_{n+2} \simeq p'_{n+2}, p'_{n+3}$. We have thus $p'_{n+1}, p'_{n+2} \simeq p'_{n+3}, p'_{n+4}$. In the same way each two pairs of points of P' whether or not they have a point in common can be proved to be congruent. The set P' is thus equilateral. In particular each subset of P' consisting of $n+2$ points is equilateral. According to property (1) of the R_n this contradicts the assumption that each $n+2$ points of S' are congruent with $n+2$ points of the R_n. The hypothesis leads thus to a contradiction and this completes the proof of the second fundamental theorem.

3. Metrical characterization of the euclidean spaces. In this last part we deal with subsets of semi-metrical spaces. To each two points p, q there corresponds a real number $pq = qp > 0$ if $p \neq q$ whereas $pp = 0$. For each k points $p_1, p_2, \cdots, p_k$ of a semi-metrical space we denote by $D(p_1, p_2, \cdots, p_k)$ the symmetric determinant

$$\begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & (p_1p_2)^2 & \cdots & (p_1p_k)^2 \\ 1 & (p_2p_1)^2 & 0 & \cdots & (p_2p_k)^2 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & (p_kp_1)^2 & (p_kp_2)^2 & \cdots & 0 \end{vmatrix}$$

which we also shall denote symbolically by

$$\begin{vmatrix} 0 & 1 \\ 1 & (p_ip_j)^2 \end{vmatrix} \qquad (i, j = 1, 2, \cdots, k).$$

The importance of this determinant lies in the fact that if $p_1, p_2, \cdots, p_k$ are k points of a euclidean space the volume $v(p_1, p_2, \cdots, p_k)$ of the $(k-1)$-dimensional simplex determined by these k points is given by the *volume formula*

$$v^2(p_1, p_2, \cdots, p_k) = \frac{(-1)^k}{(k-1)!\,2^{2k-1}} \cdot D(p_1, p_2, \cdots, p_k).$$

Evidently the determinant of k points is *congruence invariant*, i. e. if $p'_1, p'_2, \cdots, p'_k$ are congruent with $p_1, p_2, \cdots, p_k$ then

$$D(p_1, p_2, \cdots, p_k) = D(p'_1, p'_2, \cdots, p'_k).$$

We denote by $D(p_1, p_2, \cdots, p_{k-2}, p_{k-1}, p_k; x)$ the function of x which we obtain by substituting x in place of the terms $(p_{k-1}p_k)^2$ and $(p_kp_{k-1})^2$ in the determinant $D(p_1, p_2, \cdots, p_k)$. We may symbolize this function also by

$$D(p_1, \cdots, p_{k-2}, p_{k-1}, p_k; x) = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & (p_ip_j)^2 & (p_ip_{k-1})^2 & (p_ip_k)^2 \\ 1 & (p_{k-1}p_j)^2 & 0 & x \\ 1 & (p_kp_j)^2 & x & 0 \end{vmatrix}$$

$$(i, j = 1, 2, \cdots, k-2).$$

Remark a. *The coefficient of x^2 in the quadratic polynomial $D(p_1, \cdots, p_k; x)$ is $-D(p_1, p_2, \cdots, p_{k-2})$.* This follows immediately if we develop D according to Laplace.

Remark b. If $p_1, \cdots, p_{k-2} \simeq p'_1, \cdots, p'_{k-2}$

$$p_ip_{k-1} = p'_ip'_{k-1}, \qquad p_ip_k = p'_ip'_k \qquad (i = 1, 2, \cdots, k-2)$$

then $$D(p_1, \cdots, p_{k-2}, p_{k-1}, p_k; x) \equiv D(p'_1, \cdots, p'_{k-2}, p'_{k-1}, p'_k; x).$$

The main result of this third part will be the

THIRD FUNDAMENTAL THEOREM. *In order that $n+2$ points $p'_1, p'_2, \cdots, p'_{n+2}$ of a semi-metrical space each $n+1$ of which are congruent with $n+1$ points of the R_n be congruent with $n+2$ independent points of the R_{n+1} or with $n+2$ points of the R_n it is necessary and sufficient that*

$$\text{sign } D(p'_1, p'_2, \cdots, p'_{n+2}) = (-1)^n \text{ or } D(p'_1, p'_2, \cdots, p'_{n+2}) = 0$$

respectively.

If $p'_1, p'_2, \cdots, p'_{n+2}$ are congruent with $n+2$ points $p_1, p_2, \cdots, p_{n+2}$ of the R_{n+1} then $D(p'_1, p'_2, \cdots, p'_{n+2}) = D(p_1, p_2, \cdots, p_{n+2})$. According to the volume formula $(-1)^{n+2}$. $D(p_1, p_2, \cdots, p_{n+2})$ is identical with the square of the volume of the $(n+1)$-dimensional simplex determined by the points $p_1, p_2, \cdots, p_{n+2}$ (if we neglect a positive factor). As the square of this volume is not negative, and is zero if and only if the $n+2$ points lie in

a n-dimensional plane of the R_{n+1}, it follows that the condition of the theorem is *necessary*.

In order to prove that the condition is *sufficient* let us assume that $p'_1, p'_2, \cdots, p'_{n+2}$ are $n+2$ points each $n+1$ of which are congruent with $n+1$ points of the R_n and whose determinant has the sign $(-1)^n$ or is zero. We have to prove the existence of $n+2$ congruent points of the R_{n+1} which are independent or dependent respectively.

This proposition is trivial in the case $n=0$. For $D(p'_1, p'_2) = 2(p'_1 p'_2)^2$. Hence p'_1 and p'_2 are identical if and only if $D(p'_1, p'_2) = 0$. In this case they are congruent with two points of the R_0 (consisting of one point). If sign $D(p'_1, p'_2) = (-1)^0$, i. e. is positive, then $p'_1 p'_2$ is a real positive number and p'_1 and p'_2 are congruent with two points of the R_1 which are independent (i. e. not identical).

We prove our proposition therefore under the inductive hypothesis of its truth for $n+1$ points, i. e. we make the assumption: If $n+1$ points, each n of which are congruent with n points of the R_{n-1}, have a determinant of the sign $(-1)^{n-1}$ or 0 then they are congruent with $n+2$ independent points of an n-dimensional plane of the R_{n+1} or with $n+2$ points of a $(n-2)$-dimensional plane of the R_n respectively. We deduce first of all some conclusions from this assumption.

Lemma 4. *If $p_1, p_2, \cdots, p_n, p_{n+1}, p_{n+2}$ lie in an n-dimensional plane of the R_{n+1} and $p_1, p_2, \cdots, p_n$ do not lie in an $(n-2)$-dimensional plane then the equation*

$$D(p_1, p_2, \cdots, p_n, p_{n+1}, p_{n+2}; x) = 0$$

has the double root $x = (p_{n+1} p_{n+2})^2$.

Let us denote the polynomial $D(p_1, \cdots, p_{n+2}; x)$ by $D(x)$. If we substitute the value $x = (p_{n+1} p_{n+2})^2$ in $D(x)$ it becomes $D(p_1, \cdots, p_n, p_{n+1}, p_{n+2})$. As the $n+2$ points $p_1, \cdots, p_{n+2}$ lie in an n-dimensional plane Π their determinant equals 0, according to the necessity of the condition of the third fundamental theorem which already has been proved. Hence $x = (p_{n+1} p_{n+2})^2$ is a root of the equation $D(x) = 0$. In order to prove the lemma we have thus merely to prove that the equation $D(x) = 0$ has a double root. In order to prove this we remark first of all that the R_{n+1} is divided by the n-dimensional plane Π into two open parts which may be distinguished as the left part and the right part. We determine a sequence of points of the left part converging towards the point p_{n+1}, say p^k_{n+1} $(k = 1, 2, \cdots$ ad inf.), and a sequence of points of the right part converging towards p_{n+2}, say p^k_{n+2} $(k = 1, 2, \cdots$ ad inf.). We consider for each integer

k the polynomial $D(p_1, \cdots, p_n, p^k{}_{n+1}, p^k{}_{n+2}; x)$ which may be denoted by $D^k(x)$. For each k one root of the equation $D^k(x) = 0$ is $x_k = (p^k{}_{n+1} p^k{}_{n+2})^2$. If we denote by $\bar{p}^k{}_{n+2}$ the mirror image of the point $p^k{}_{n+2}$ in Π, then, in consequence of remark *b*, we have

$$D(p_1, \cdots, p_n, p^k{}_{n+1}, p^k{}_{n+2}) \equiv D(p_1, \cdots, p_n, p^k{}_{n+1}, \bar{p}^k{}_{n+2}).$$

Hence each zero of the second polynomial is also a root of the equation $D^k(x) = 0$. A zero of the second polynomial is $\bar{x}_k = (p^k{}_{n+1} \bar{p}^k{}_{n+2})^2$. As $p^k{}_{n+1} p^k{}_{n+2} \neq p^k{}_{n+1} \bar{p}^k{}_{n+2}$ we know then, two roots of the equation $D^k(x) = 0$. This equation does not vanish identically as x^2 has, according to remark *a*, the coefficient $D(p_1, \cdots, p_n)$ which, is $\neq 0$ as $p_1, \cdots, p_n$ are supposed not to be in a $(n-2)$-dimensional plane. The values $x_k = (p^k{}_{n+1} p^k{}_{n+2})^2$ and $\bar{x}_k = (p^k{}_{n+1} \bar{p}^k{}_{n+2})^2$ are thus the two roots of the equation $D^k(x) = 0$. As the points $p^k{}_{n+2}$ tend towards the point p_{n+2} of Π their images in Π, i. e. the points $\bar{p}^k{}_{n+2}$ tend toward the same point. It follows that $\lim (x_k - \bar{x}_k) = 0$. If we denote by D_k the discriminant of the equation $D^k(x) = 0$ it follows that $\lim_{k=\infty} D_k = 0$. As the points $p^k{}_{n+1}$ and $p^k{}_{n+2}$ tend toward p_{n+1} and p_{n+2} respectively it follows that the polynomials $D^k(x)$ tend toward the polynomial $D(x)$. As the coefficient of x^2 in $D(x)$ equals $D(p_1, p_2, \cdots, p_n)$ and hence is not zero it follows that the discriminant of $D(x) = 0$ is the limit of the discriminants D_k. This limit, as we saw, is 0. Hence $D(x) = 0$ is a not identically vanishing quadratic equation whose discriminant equals zero and, therefore, has a double root. Lemma 4 is thus proved.

LEMMA 5. *If $p'_1, \cdots, p'_{n+2}$ is a pseudo-euclidean $(n+2)$-tuple then sign $D(p'_1, \cdots, p'_{n+2}) = (-1)^{n+1}$.*

There exist, as we know, $n+2$ points $p_1, \cdots, p_n, p_{n+1}, p_{n+2}$ in the R_n such that

$$p_1, \cdots, p_n, p_{n+1} \simeq p'_1, \cdots, p'_n, p'_{n+1}$$
$$p_1, \cdots, p_n, p_{n+2} \simeq p'_1, \cdots, p'_n, p'_{n+2}$$
$$p_{n+1} p_{n+2} \neq p'_{n+1} p'_{n+2}.$$

According to remark *b* it follows from these relations that

$$D(p'_1, p'_2, \cdots, p'_n, p'_{n+1}, p'_{n+2}; x) \equiv D(p_1, p_2, \cdots, p_n, p_{n+1}, p_{n+2}; x).$$

These polynomials will be denoted by $D'(x)$ and $D(x)$ respectively. The points $p_1, \cdots, p_n$ are congruent with n points of a pseudo-euclidean $(n+2)$-tuple and hence, according to theorem II on pseudo-euclidean sets, do not lie in a $(n-2)$-dimensional plane. According to lemma 4 the equation

$D(x)=0$ has thus the double root $(p_{n+1}p_{n+2})^2$. This value is thus a double root of the equation $D'(x)=0$, also. For each other value the quadratic polynomial $D'(x)\equiv D(x)$ has the same sign. For large values of x the sign of $D(x)$ is identical with the sign of the coefficient of x^2 in $D(x)$. This coefficient according to remark a equals $-D(p_1,\cdots,p_n)$. As the points $p_1,\cdots,p_n$ do not lie in a $(n-2)$-dimensional plane it follows from the necessity of the condition of the third fundamental theorem which already has been proved that sign $D(p_1,\cdots,p_n)=(-1)^n$. The sign of $D(x)$ and $D'(x)$ for each value of x that is different from $(p_{n+1}p_{n+2})^2$ is thus $(-1)^{n+1}$. As $p'_{n+1}p'_{n+2}\neq p_{n+1}p_{n+2}$ the lemma 5 is proved.

We are now in the position to prove the sufficiency of the conditions of the third fundamental theorem. Let us assume that $p'_1,\cdots,p'_{n+2}$ are $n+2$ points each $n+1$ of which are congruent to $n+1$ points of the R_n and such that

$$\text{sign } D(p'_1,\cdots,p'_{n+2}) = (-1)^n \text{ or } 0.$$

We have to prove that there exist $n+2$ congruent points in the R_{n+1} independent or lying in a n-dimensional plane respectively.

According to the hypothesis on $D(p'_1,\cdots,p'_{n+2})$ it is impossible by the lemma that the $n+2$ given points form a pseudo-euclidean $(n+2)$-tuple. If each $n+1$ of our $n+2$ points are congruent with $n+1$ points of the R_{n-1} then the $n+2$ points are thus congruent with $n+2$ points of the R_{n-1} and a fortiori with $n+2$ points of the R_{n+1}. (This case evidently can only happen if $D(p_1,\cdots,p_{n+2})=0$.)

We may thus asssume that $n+1$ of the $n+2$ points are not congruent with $n+1$ points of the R_{n-1}, let us say $p'_1,\cdots,p'_{n+1}$. There exist $n+1$ congruent points $p_1,\cdots,p_{n+1}$ in an n-dimensional plane Π of the R_{n+1}. The points $p_1,\cdots,p_n$ lie in a $(n-1)$-dimensional plane Π^* but not in a $(n-2)$-dimensional plane. As each $n+1$ of our $n+2$ points are congruent with $n+1$ points of the R_n there exists at least one point p in Π such that

$$p_1,p_2,\cdots,p_n,p \asymp p'_1,p'_2,\cdots,p'_n,p'_{n+2}.$$

If we rotate p in the R_{n+1} around the $(n-1)$-dimensional plane Π^* the path of p is a circle perpendicular to Π^*. The circle has the radius O, i. e. degenerates in a single point, if and only if the point lies in Π^*, and hence if and only if the points $p'_1,\cdots,p'_n,p'_{n+2}$ are congruent with $n+1$ points of a R_{n-1}. If this is not the case then the circle has exactly two points $\bar{p}_{n+2}$ and $\bar{\bar{p}}_{n+2}$ in common with Π. As p_{n+1} does not lie in Π^* we have $p_{n+1}\bar{p}_{n+2}\neq p_{n+1}\bar{\bar{p}}_{n+2}$ and we assume the notation such that

$$p_{n+1}\bar{p}_{n+2} < p_{n+1}\bar{\bar{p}}_{n+2}.$$

Let us now assume that $D(p'_1, \cdots, p'_{n+2}) = 0$ and prove that there exist $n+2$ points congruent with $p'_1, \cdots, p'_{n+2}$ in the n-dimensional plane Π. We distinguish two cases. First it may be that the points $p'_1, \cdots, p'_n, p'_{n+2}$ are congruent with $n+1$ points of the R_{n-1}. We form then the above mentioned points $p_1, \cdots, p_n, p_{n+1}, p$ and know that p lies in Π^*. Thus $p_1, \cdots, p_n, p_{n+1}, p$ lie in the n-dimensional plane Π whereas $p_1, \cdots, p_n$ do not lie in a $(n-2)$-dimensional plane. According to lemma 4 the equation $D(p_1, \cdots, p_n, p_{n+1}, p; x) = 0$ has the double root $x = (p_{n+1}p)^2$. According to remark b the polynomials $D(p'_1, \cdots, p'_n, p'_{n+1}, p'_{n+2}; x)$ and $D(p_1, \cdots, p_n, p_{n+1}, p_{n+2}; x)$ are identical. Hence $x = (p_{n+1}p)^2$ is also the double root of the first of these polynomials which may be denoted by $D'(x)$. From the hypothesis that $D(p'_1, \cdots, p'_{n+1}) = 0$ it follows that $x = (p'_n p'_{n+1})^2$ is a root of the equation $D'(x) = 0$. The equation $D(x) = 0$ does not vanish indentically as the coefficient of x^2 is $-D(p'_1, \cdots, p'_n) = -D(p_1, \cdots, p_n)$ and this is not zero as the points $p_1, \cdots, p_n$ do not lie in a $(n-2)$-dimensional plane. It follows that $p'_{n+1}p'_{n+2} = p_{n+1}p$ and hence the $n+2$ points $p'_1, \cdots, p'_{n+1}, p'_{n+2}$ are congruent with the $n+2$ points $p_1, \cdots, p_{n+1}, p$ of the n-dimensional plane Π.

The second case is that the points $p'_1, \cdots, p'_{n+2}$ are not congruent with $n+1$ points of the R_{n-1}. In this case $x = (p_{n+1}\bar{p}_{n+2})^2$ and $x = (p_{n+1}\bar{\bar{p}}_{n+2})^2$ are two solutions of the equation $D(p_1, \cdots, p_n, p_{n+1}, p; x) = 0$. For if we substitute these two values we get the determinant of $n+2$ points of the n-dimensional plane Π, viz. $D(p_1, \cdots, p_n, p_{n+1}, \bar{p}_{n+2})$ and $D(p_1, \cdots, p_n, p_{n+1}, \bar{\bar{p}}_{n+2})$ respectively, which are both equal to zero. According to remark b the above mentioned equation is identical with the equation $D(p'_1, \cdots, p'_n, p'_{n+1}, p'_{n+2}; x) = 0$. A root of this equation is $x = (p'_{n+1}p'_{n+2})^2$, for if we substitute this value we get $D(p'_1, \cdots, p'_{n+2})$ which is zero, according to the hypothesis. It follows that $p'_{n+1}p'_{n+2}$ equals either $p_{n+1}p_{n+2}$ or $p_{n+1}\bar{p}_{n+2}$. Hence the $n+2$ points $p'_1, \cdots, p'_{n+1}, p'_{n+2}$ are congruent with either $p_1, \cdots, p_{n+1}, p_{n+2}$ or with $p_1, \cdots, p_{n+1}\bar{\bar{p}}_{n+2}$ and thus in either case, congruent with $n+2$ points of the n-dimensional plane Π.

Let us assume now that sign $D(p'_1, \cdots, p'_{n+2}) = (-1)^n$ and prove that there exist $n+2$ independent points of the R_{n+1} which are congruent with $p'_1, \cdots, p'_{n+2}$. We form again the points $p_1, \cdots, p_{n+1}, \bar{p}_{n+2}, \bar{\bar{p}}_{n+2}$ and consider the polynomial $D(p'_1, \cdots, p'_{n+1}, p'_{n+2}; x)$ which may be denoted by $D'(x)$. According to remark b the three polynomials

$$D'(x),\ D(p_1, \cdots, p_{n+1}, \bar{p}_{n+2}; x),\quad D(p_1, \cdots, p_{n+1}, \bar{\bar{p}}_{n+2}; x)$$

are identical. A zero of the second is $x = (p_{n+1}\bar{p}_{n+2})^2$, a zero of the third is $x = (p_{n+1}\bar{\bar{p}}_{n+2})^2$. These two numbers, the first of which has been assumed to be smaller than the second, are thus roots of the equation $D'(x) = 0$, also. The coefficient of x^2 in $D'(x)$ is $-D(p'_1, \cdots, p'_n)$ and thus equals $-D(p_1, \cdots, p_n)$. As the n points $p_1, \cdots, p_n$ do not lie in a $(n-2)$-dimensional plane their determinant is not zero and has the sign $(-1)^n$. The coefficient of x^2 in $D'(x)$ has, therefore, the sign $(-1)^{n+1}$ and this is the sign of $D'(x)$ for large values of x. As $D'(x)$ is a quadratic polynomial with the two different zeros $(p_{n+1}\bar{p}_{n+2})^2$ and $(p_{n+1}\bar{\bar{p}}_{n+2})^2$ we conclude:

For $x \leqq (p_{n+1}\bar{p}_{n+2})^2$, sign $D'(x) = (-1)^{n+1}$ or 0.

For $(p_{n+1}\bar{p}_{n+2})^2 < x < (p_{n+1}\bar{\bar{p}}_{n+2})^2$, sign $D'(x) = (-1)^n$.

For $x \geqq (p_{n+1}\bar{\bar{p}}_{n+2})^2$, sign $D'(x) = (-1)^{n+1}$ or 0.

It follows thus from sign $D'(x) = (-1)^n$ that $(p_{n+1}\bar{p}_{n+2})^2 < x < (p_{n+1}p_{n+2})$. We have supposed that sign $D(p'_1, \cdots, p'_{n+1}, p'_{n+2}) = (-1)^n$. That means that $D'(x)$ for $x = (p'_{n+1}p'_{n+2})^2$ has the sign $(-1)^n$. We have thus proved:

(*) $$(p_{n+1}\bar{p}_{n+2})^2 < (p'_{n+1}p'_{n+2})^2 < (p_{n+1}\bar{\bar{p}}_{n+2})^2$$

and hence

$$p_{n+1}\bar{p}_{n+2} < p'_{n+1}p'_{n+2} < p_{n+1}\bar{\bar{p}}_{n+2}.$$

Let us now consider the above mentioned circle obtained by the rotation of p around the $(n-1)$-dimensional plane Π^* and perpendicular to Π. The nearest point to p_{n+1} on this circle is the point $\bar{p}_{n+2}$. If we move on the circle from this point toward the point $\bar{\bar{p}}_{n+2}$ then the distance from p_{n+1} increases continuously till it attains its maximum in the point $\bar{\bar{p}}_{n+2}$ which is the remotest point from p_{n+1} on the circle. In consequence of the inequality (*) there exists a point on the circle which we shall call p_{n+2}, such that $p_{n+1}p_{n+2} = p'_{n+1}p'_{n+2}$. This point is identical neither with $\bar{p}_{n+2}$ not $\bar{\bar{p}}_{n+2}$ and hence does not lie in the n-dimensional plane Π. The $n+2$ points $p_1, p_2, \cdots p_{n+1}, p_{n+2}$ are thus $n+2$ independent points of the R_{n+1} congruent with the $n+2$ given points $p'_1, \cdots, p'_{n+1}, p'_{n+2}$. The proof of the third fundamental theorem is thus completed.

It follows from the proof that under the hypothesis, sign $D(p'_1, \cdots, p'_{n+2}) = (-1)^n$ there exist $n+2$ independent points of the R_{n+1} congruent with the points $p'_1, \cdots, p'_{n+2}$ if we merely suppose that *two* of the $(n+1)$-tuples which have in common n points not congruent with n points of the R_{n-2} are congruent with $(n+1)$-tuples of the R_n.

We have proved lemma 5 for pseudo-euclidean $(n+2)$-tuples under the

hypothesis that the third fundamental theorem is true for $(n+1)$-tuples. From the third fundamental theorem we may conclude, therefore, the following

THEOREM 3 ON PSEUDO-EUCLIDEAN SETS. *The sign of the determinant of a pseudo-euclidean* $(n+3)$*-tuple is* $(-1)^n$.

A set consisting of $n+3$ points each $n+2$ of which are congruent with $n+2$ points of the R_n has been called pseudo-euclidean if it is not congruent with a subset of the R_n. From theorem 3 it follows immediately that such a pseudo-euclidean set is not congruent with a subset of the R_{n+1} or the R_{n+2} either. We see thus, that *a pseudo-euclidean set is not congruent with a subset of any euclidean space or even of Hilbert space.* Another corollary of theorem 3 is that *no proper subset of a pseudo-euclidean set is pseudo-euclidean.*

An immediate consequence of the second and third fundamental theorem is the following

METRICAL CHARACTERISATION OF EUCLIDEAN SETS. *In order that a semi-metrical space S be congruent with a subset of the* R_n *it is necessary and sufficient that, if S contains more than* $n+3$ *points, for each integer* $k \leqq n+1$ *and for each k-tuple of points* $p_1, \cdots, p_k$ *of S*

$$\text{sign } D(p_1, \cdots, p_k) = (-1)^k \text{ or } 0$$

and that the determinant of each $n+2$ *points of S vanishes. If S contains exactly* $n+3$ *points* $p_1, \cdots, p_{n+3}$ *it is necessary and sufficient for the congruence of S with a subset of the* R_n *that besides these conditions* $D(p_1, \cdots, p_{n+3}) = 0$.

Some further details whose proofs have been given in other papers may be mentioned in terminating this article. As to the pseudo-euclidean quadruples it has been proved * that they are characterized among the semi-metrical spaces consisting of four distinct points p_1, p_2, p_3, p_4 by the relations

$$p_1p_2 = p_3p_4, \qquad p_1p_3 = p_2p_4, \qquad p_1p_4 = p_2p_3 = p_1p_2 + p_1p_3.$$

It has been shown,† furthermore, that the facts concerning pseudo-euclidean quadruples as well as concerning the congruence order 4 and the quasi congruence order 3 of the R_1 are merely special cases of theorems of abstract groups.

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* Cp. Menger, *Mathematische Annalen*, Vol. 100, p. 126.

† Cp. Menger, *Mathematische Zeitschrift*, Vol. 33, p. 126.

The theory of pseudo-euclidean $(n+3)$-tuples in general is connected * with the theory of certain Cremona transformations of the R_n. In the case $n=2$ the theory of the isogonal transformations of the plane is applied. It follows from facts concerning isogonal transformations in particular † that in a pseudo-euclidean quintupel not only no three points are congruent with three points of the straight line but, furthermore, that no four points are congruent with four points of a circle.

In order that a metrical space S be congruent with a subset of the Hilbert space it is necessary and sufficient ‡ that S be separable (i. e. contains a dense subset which is denumerable) and that for each integer k and for each k points $p_1, \cdots, p_k$ of S

$$\operatorname{sgn} D(p_1, \cdots, p_k) = (-1)^k \text{ or } 0.$$

It is remarkable that the condition of separability cannot be omitted whereas it was not necessary to postulate it in the euclidean case.

The condition $\operatorname{sign} D(p_1, p_2, p_3) = -1$ or 0 is equivalent to the so called triangle-inequality. Its being valid for each three points of a semi-metrical space is thus characteristic for the metrical spaces. It has been proved § that in a complete metrical space which is convex ¶ and exteriorly convex there exists for each two distinct points p and q a subset containing p and q which is congruent with the straight line. It follows ‖ that among the semi-metrical spaces satisfying the conditions which are characteristic for the subsets of the R_n the R_n itself is characterized by the properties of being complete, convex, exteriorly convex and containing $n+1$ points which are not congruent with $n+1$ points of the R_{n-1}.

The R_1 has been proved ** to be the only metrical space which is convex, exteriorly convex, complete, and has the quasi-congruence order 3.

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* Cp. *loc. cit.* *, p. 126.

† Cp. *loc. cit.* *, p. 127.

‡ Cp. Menger, *Wien. akad. Anzeiger*, 1928.

§ Cp. *loc. cit.* *, p. 87.

¶ Cp. *loc. cit.* *, p. 81. A metrical space is called convex or exteriorly convex provided that it contains for each two distinct points p and q a point s distinct from p and q and such that $ps + sq = pq$ or $pq + qs = ps$ respectively.

‖ Cp. *loc. cit.* *, p. 140.

** Cp. Menger, *Wien. akad. Anzeiger*, 1931.

NOTE ON THE GREEN FUNCTION OF A STAR-SHAPED THREE DIMENSIONAL REGION.*

By J. J. GERGEN.

1. *Introduction.* The object in this note is to prove the following theorem.

THEOREM. *Suppose that the region† D in the (x, y, z) space is star-shaped‡ with regard to the origin O, and that $g(x, y, z)$ is the Green function§ of D with pole at O; then at every point in $D - O$ the square of the gradient*

$$(\nabla g)^2 = g_x^2 + g_y^2 + g_z^2$$

of g satisfies the condition

$$(\nabla g)^2 \geq g^2/r^2 > 0 \tag{1.1}$$

* Preliminary report presented to the American Mathematical Society, December 30, 1930. The numerals in bold-faced type in this paper refer to the following treatises and papers:

1. Carathéodory, C., *Vorlesungen über Reele Funktionen*, Leipzig (1927).
2. Gergen, J. J., "Mapping of a General Type of Three Dimensional Region on a Sphere," *American Journal of Mathematics*, Vol. 52 (1930), pp. 197-224.
3. Kellogg, O. D., "On the Classical Dirichlet Problem for General Domains," *Proceedings of the National Academy of Sciences*, Vol. 12 (1926), pp. 397-406.
4. ————, "Recent Progress with the Dirichlet Problem," *Bulletin of the American Mathematical Society*, Vol. 32 (1926), pp. 601-625.
5. ————, "On the Derivatives of Harmonic Functions on the Boundary," *Transactions of the American Mathematical Society*, Vol. 33 (1931), pp. 486-510.
6. Liapounoff, M. A., "Sur certaines questions qui se rattachent au problème de Dirichlet," *Journal de Mathématiques (Liouville)*, Vol. 40 (1898), pp. 241-311.
7. Neumann, E. R., *Studien über die Methoden von C. Neumann und G. Robin fur Lösung der beiden Randwertaufgaben der Potentialtheorie*, Leipzig (1905).
8. Osgood, W. F., *Lehrbuch der Funktionentheorie*, Vol. 1, Berlin (1907).

† By a region is meant an open continuum.

‡ A region or, in general, any set of points E is star-shaped with regard to a point O when, and only when, each line segment joining O to a point in E lies entirely in E.

§ The Green function of an arbitrary star-shaped region may not exist in the ordinary sense. In the theorem we understand by g the generalized Green function of D. The generalized Green function is, of course, identical with the ordinary one in case the latter exists. For the definitions and properties of the generalized Green function, and also for a discussion and set of references on the Dirichlet problem, see Kellogg **3** and **4**. In these papers Kellogg considers, in general, only regions whose boundaries are bounded point sets; but no such restriction is necessary in his definition and proof of the uniqueness of the generalized Green function.

where $r=(x^2+y^2+z^2)^{\frac{1}{2}}$, *the inequality holding throughout* $D-O$ *unless* $g-1/r$ *vanishes identically there.** *Furthermore, for every positive constant* c, *the region* † *determined by the inequality* $g>c$ *is star-shaped with regard to* O.

This theorem has, as corollary, that the Green function corresponding to a convex, three dimensional region, and arbitrarily assigned pole has a non-vanishing gradient. This fact was conjectured and mentioned to the author by Professor Morse; and it was in an attempt to prove this conjecture tenable that the theorem was established.

In connection with these results it should perhaps be recalled that, in contrast with the corresponding situation in two dimensions,‡ the gradient of the Green function of a simply-connected three dimensional region may vanish at points in the region.§ The importance of these results is, then, that they furnish a partial answer to the general question as to what conditions on a region and the position of the pole are sufficient to insure the absence of equilibrium points of the Green function corresponding to that region and pole. The theorem gives us, of course, more information about the Green function of a star-shaped region than that its gradient is non-vanishing. Incidentally, the inequality (1. 1) is the best possible, in the sense that, if D is the whole of space, so that $g=1/r$, we have

$$(\nabla g)^2=g^2/r^2$$

throughout $D-O$.

The proof of the theorem depends largely on three facts:

(i) If the frontier d of D is a bounded, regular, analytic surface, then each of the partial derivatives g_x, g_y, g_z, of g coincides in D with a function which is continuous in $D+d-O$.

(ii) The function

$$u(x,y,z)=r\partial g/\partial r=xg_x+yg_y+zg_z$$

is harmonic in $D-O$.

(iii) If $f(Q)$ is defined ¶ and lower semi-continuous for Q on a sphere σ, and if

$$f(Q)\geq R,$$

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* In this exceptional case D is, of course, equivalent to the whole of space, in the sense that its generalized Green function is the same as the Green function of the latter.

† This region, by definition, contains the point O at which $g=\infty$.

‡ Cf. Osgood, **8**, pp. 588-590.

§ For an example illustrating this fact, see Gergen, **2**, pp. 198-200.

¶ We admit here as a possible value of f the value ∞.

where R is independent of Q, then a sequence $\{\phi_n(Q)\}$, $n=1, 2, \cdots$, of functions $\phi_n(Q)$, continuous on σ, can be so defined that

$$R \leq \phi_1(Q) \leq \phi_2(Q) \leq \cdots \leq \phi_n(Q) \cdots \leq f(Q), \tag{1.2}$$

$$\lim_{n\to\infty} \phi_n(Q) = f(Q). \tag{1.3}$$

The truth of (i)* and (iii)† is well known, while (ii) can be verified by differentiation.

2.1. *Lemmas.* To establish the theorem we first prove two lemmas relative to D, g, and u.

LEMMA 1. *If D is bounded, and star-shaped with regard to O and if the boundary d of D is a regular, analytic surface, then*

$$u(x, y, z) \leq -g(x, y, z) \tag{2.11}$$

in $D-O$.‡

To prove this, consider the function $w(x, y, z)$ defined as

$$w = u + g$$

in $D-O$. By (ii), u is harmonic in $D-O$, and thus, since g is harmonic there, w has the same property. Further, by (i), u is continuous in $D+d-O$, when defined on d by means of its limiting values. Hence, since g vanishes continuously on d, w is continuous in $D+d-O$, when properly defined on d. Finally, since g has the form

$$g = 1/r + v(x, y, z),$$

where v is harmonic in D, we have

$$w = v + xv_x + yv_y + zv_z;$$

and it follows that w is harmonic at O, when defined as $v(0, 0, 0)$ there. Thus w is harmonic in D and continuous in $D+d$. To show, then, that (2.11) holds, it is enough to prove that w is never positive on d.

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* Cf. Liapounoff, **6**, pp. 305-311, and Neumann, **7**, p. 178. Liapounoff proves that the partial derivatives of g have the property stated for a much more general type of boundary, provided also that the Neumann principle is applicable to the boundary. Neumann proves that the principle in question is applicable to boundaries of a very general character, in particular, to boundaries of the type specified above. For a statement of the Neumann principle, see Liapounoff, p. 268. See also Kellogg, **5**. Kellogg considers the boundary values of the partial derivatives of all orders of harmonic functions. He obtains results which include (i) as a particular case.

† For a proof of (iii) see Carathéodory, **1**.

‡ It is classical that the ordinary Green function exists for regions of this type.

Let $P_1(x_1, y_1, z_1)$ be any point on d. Then, since D is star-shaped with regard to O, and d is an analytic surface, the segment OP_1 lies entirely in D, except for its extremity P_1. Thus we have

$$w(x_1, y_1, z_1) = \lim_{\eta \to 1-0} w(\eta x_1, \eta y_1, \eta z_1).$$

But this is evidently

$$w(x_1, y_1, z_1) = \lim_{\eta \to 1-0} u(\eta x_1, \eta y_1, \eta z_1),$$

since g vanishes on d. Accordingly,

$$w(x_1, y_1, z_1) = \lim_{\eta \to 1-0} \left\{ \eta \mid OP_1 \mid [\partial g/\partial r]^{\substack{x=\eta x_1 \\ y=\eta y_1 \\ z=\eta z_1}} \right\}$$

$$= \mid OP_1 \mid \lim_{\eta \to 1-0} \left\{ [\partial g/\partial r]^{\substack{x=\eta x_1 \\ y=\eta y_1 \\ z=\eta z_1}} \right\}.$$

Now, since the limit on the right here exists, and since g is positive in D and vanishes continuously on d, we must have $w(x_1, y_1, z_1) \leq 0$. This completes the proof of the lemma.

2.2. LEMMA 2. *An infinite sequence* $\{D_n\}$, $n = 1, 2, \cdots$, *of bounded regions* D_n *can be so constructed that*

(a) $$D_1 \subset D_2 \subset \cdots \subset D_n \cdots \subset D,$$

(b) $$D_1 + D_2 + \cdots + D_n + \cdots = D,$$

(c) *the boundary* d_n *of* D_n *is a regular, analytic surface,*

(d) D_n *is star-shaped with regard to* O.

The proof rests on (iii). Consider a point Q on the unit sphere σ about O. The ray $\gamma(Q)$ through Q, issuing from O, either fails to pierce d or else, since d is closed, there is a first point, starting from O, of intersection of $\gamma(Q)$ with d. At each point Q of σ, then, let $f(Q)$ be defined as the distance from O to this first point of intersection if $\gamma(Q)$ pierces d, and as ∞ otherwise.

The function f, we observe, is lower semi-continuous on σ, for d is closed. Moreover, since O is a point of D and D is open, there is a number $R > 0$ such that

$$f(Q) \geq R$$

for Q on σ. Hence, by (iii), we can find a sequence $\{\phi_n(Q)\}$, $n = 1, 2, \cdots$, of functions $\phi_n(Q) = \phi_n(x, y, z)$, continuous on σ, such that (1.2) and (1.3) hold.

Now, since $\phi_n(Q)$ is continuous, there exists a function

$$\psi_n(P) = \psi_n(x, y, z)$$

which is harmonic in the interior Σ of σ, continuous in $\sigma + \Sigma$, and which assumes the value $n\phi_n(Q)/(n+1)$ at each point Q of σ.

By means of ψ_n we define D_n. We choose $r_n < 1$ so near to one that, if Q is any point on σ and P the point of intersection of OQ with the sphere σ_n of radius r_n about O,

$$|\psi_n(P) - \psi_n(Q)| < R/(4n)^2; \tag{2.21}$$

and we define D_n as the set of points determined by the inequality

$$r < \psi_n(xr_n/r, yr_n/r, zr_n/r).$$

A point P belongs to D_n, then, when, and only when,

$$|OP| < \psi_n(P') \tag{2.22}$$

where P' is the point of intersection of OP, or OP extended, with σ_n. The set D_n is, we observe, not null, for, if $P(x, y, z)$ be any point on σ_n,

$$\begin{aligned}\psi_n(P) &> \psi_n(x/r_n, y/r_n, z/r_n) - R/(4n)^2 \\ &= n\phi_n(x/r_n, y/r_n, z/r_n)/(n+1) - R/(4n)^2 \\ &> 7R/16,\end{aligned}$$

so that D_n contains all the points in the interior of the sphere of radius $7R/16$ about O.

Now plainly D_n is bounded, open, and star-shaped with regard to O. Moreover, since the boundary d_n of D_n is represented by the equation

$$h_n(x, y, z) = r - \psi_n(xr_n/r, yr_n/r, zr_n/r) = 0,$$

and since $\partial h_n/\partial r = 1$ and ψ_n is harmonic and therefore analytic in Σ, d_n is a regular analytic surface. It remains, then, to prove that (a) and (b) hold.

Consider (a). Let P be any point of D_n, and P_k, Q the points of intersection of OP, or OP extended, with σ_k and σ, respectively. By (1.2), (2.21) and (2.22),

$$|OP| < \psi_n(P_n) < n\phi_n(Q)/(n+1) + R/(4n)^2 < \phi_n(Q),$$

whence

$$|OP| < f(Q), \tag{2.23}$$

and

$$\begin{aligned}|OP| &< n\phi_{n+1}(Q)/(n+1) + R/(4n)^2 \\ &< (n+1)\phi_{n+1}(Q)/(n+2) - R/4(n+1)^2,\end{aligned}$$

whence

$$|OP| < \psi_{n+1}(P_{n+1}). \tag{2.24}$$

From (2. 23) it follows that D_n is contained in D, and from (2. 34) that D_n is contained in D_{n+1}.

As for (b), we have, if P is any point of D and Q the point of intersection of OP, or OP extended, with σ,

$$|OP| < f(Q),$$

since D is star-shaped with regard to O. But, denoting by P_k the point of intersection of OQ with σ_k, we have also, by (1. 3),

$$\lim_{n\to\infty} \psi_n(P_n) = \lim_{n\to\infty} \phi_n(Q) = f(P).$$

Hence, for n sufficiently large,

$$|OP| < \psi_n(P_n),$$

and so D_n contains P. This completes the proof.

3. *Proof of the theorem.* As a consequence of Lemmas 1 and 2 the proof of the theorem is almost immediate. Let $g_n(x, y, z)$ be Green's function of D_n with pole at O. Then $g_n - g$ tends to zero with $1/n$ in $D - O$,* the convergence being uniform in any closed subset of $D - O$. Moreover, because of this uniform convergence,

$$u_n = xg_{nx} + yg_{ny} + zg_{nz}$$

tends to

$$u = xg_x + yg_y + zg_z$$

in $D - O$. Accordingly, by Lemma 1,

$$u \leq -g$$

in $D - O$. But, by the reasoning of Lemma 1, $u + g$ is harmonic in D and thus

$$u < -g \tag{3. 1}$$

in $D - O$ unless

$$u + g = 0$$

throughout $D - O$. This equation is, however, equivalent to the equation

$$u + g = v + xv_x + yv_y + zv_z = 0,$$

where v is harmonic in D and given by

$$v = g - 1/r$$

* Cf. Kellogg, **3**, p. 398, and the fourth footnote of this paper.

in $D - O$; thus (3.1) holds unless

$$v(0, 0, 0) = 0.$$

This is possible only when v vanishes throughout D,* and thus (3.1) holds unless g reduces to $1/r$.

Now

(3.2) $$g > 0$$

in $D - O$, and

$$u^2 = (xg_x + yg_y + zg_z)^2 \leq r^2(\nabla g)^2$$

there; hence

$$(\nabla g)^2 \geq g^2/r^2 > 0,$$

the inequality holding unless g reduces to $1/r$. This is (1.1).

To complete the proof, we observe that (3.1) and (3.2) imply that the directional derivative of g in the direction of O at any point in $D - O$ is positive. Hence, if c is any positive constant and P_1 any point in the region $g > c$, $g(P) > c$ on the segment OP_1. Thus the region $g > c$ is star-shaped with regard to O; and the theorem is proved.

HARVARD UNIVERSITY.

* See Kellogg, **5**. We have

$$g_n = 1/r + v_n,$$

where $v_n < 0$ in D_n,

$$v = \lim_{n \to \infty} v_n.$$

Hence $v \leq 0$ in D, and therefore cannot vanish in D unless it vanishes throughout D.

A JUNCTION PROPERTY OF LOCALLY CONNECTED SETS.

By G. T. WHYBURN.

1. Let M denote a connected and locally connected metric space. In this paper it will be shown that if K is any compact subset of such a set M, there exists a subset H of M which is dense in K (i. e. every point of K either belongs to H or is a limit point of $H \cdot K$) and is the image under a *uniformly continuous* transformation of the set of all dyadic rational numbers on an interval. This proposition yields a number of important consequences, some of which will be discussed in the concluding section (§ 4) of this paper. In particular, it yields as a corollary the celebrated theorem of Hahn-Mazurkiewicz that any *compact* space M is the continuous image of an interval, and it enables one to deduce the recent Moore-Menger theorem on the arcwise connectivity of *complete* spaces M from the well known theorem that *compact* spaces M are arcwise connected.

2. LEMMA. *If F is any finite subset of M, any two points a and b of F can be joined, for every $\epsilon > 0$, by an ϵ-chain* of points containing F and each pair of successive points of which lie together in an ϵ-region* of M.*

For let the points of F be $a = p_1, p_2, \cdots, p_n = b$. Now we can join p_1 and p_2 by an ϵ-chain each pair of successive points of which lie together in an ϵ-region of M. For this purpose it is only necessary to join p_1 and p_2 by a simple chain † of $\epsilon/2$-regions and take one point from each link in this chain. Similarly we join p_2 to p_3 by such a chain of points, and then join p_3 to p_4, and so on until we reach p_n. Then clearly the sequence of points taken in the order in which they are obtained in this complete process is an ϵ-chain of points from a to b which contains F and is such that any two successive points lie together in an ϵ-region of M.

3. THEOREM. *If K is any self-compact subset of M, there exists a subset H of M which is dense in K and which is the image under a uniformly continuous transformation of the set of all dyadic rational ‡ numbers on the unit interval.*

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* An *ϵ-chain* of points joining a and b is a finite sequence $a = X_1, X_2, \cdots, X_n = b$ of points such that each pair of successive ones are at a distance $< \epsilon$ apart. A *region* in M is any connected open subset of M, and an *ϵ-region* is any region of diameter $< \epsilon$.

† See R. L. Moore, "On the Foundation of Plane Analysis Situs," *Transactions of the American Mathematical Society*, Vol. 17 (1916), Theorem 10.

‡ That is, all numbers on (0, 1) which can be written as fractions having powers

Proof. Since K is compact and metric it is therefore separable. Let $P = \sum_1^\infty p_i$ be a countable set of points in K which is dense in K and, for each integer n, let $P_n = \sum_1^n p_i$. Let $\epsilon_1, \epsilon_2, \cdots$ be a sequence of positive numbers such that $\sum_1^\infty \epsilon_i$ converges.

It follows by the Borel Theorem that K is contained in the sum of a finite number of $\epsilon_1/2$-regions of M; and if we choose one point of P in each of these regions and take an integer n_1 greater than the subscript of any of the points so chosen, then every point of K lies in an $\epsilon_1/2$-region with some point of P_{n_1}. Now let a and b be any two points of M. By the lemma, § 2, there exists an $\epsilon_1/2$-chain C^1 of points from a to b which contains P_{n_1} and is such that any two successive points lie in an $\epsilon_1/2$-region and which, furthermore, we may suppose contains exactly $2^{v_1} + 1$ points

$$a = X_0{}^1, X_1{}^1, \cdots, X^1{}_2{}^{v_1} = b.$$

Similarly, with the aid of the Borel Theorem, it follows that there exists an integer $n_2 \geq n_1$ such that every point of K lies in an $\epsilon_2/2$-region with some point of P_{n_2}. For each integer i, $0 \leq i \leq 2^{v_1}$, let $F_i{}^1$ be the set of all those points of P_{n_2} which lie together with $X_i{}^1$ in an $\epsilon_1/2$-region. Since C^1 contains P_{n_1}, it follows that every point of P_{n_2} belongs to some set $F_i{}^1$. Now for each i, there exists an ϵ_1-region $M_i{}^1$ which contains $X_i{}^1 + F_i{}^1 + X^1{}_{i+1}$, because each point of $F_i{}^1 + X^1{}_{i+1}$ lies together with $X_i{}^1$ in some $\epsilon_1/2$-region. Since the region $M_i{}^1$ is itself a connected and locally connected space, it follows by the lemma that $M_i{}^1$ contains an $\epsilon_2/2$-chain $C_i{}^1$ of points from $X_i{}^1$ to $X^1{}_{i+1}$ which contains $F_i{}^1$ and is such that any two successive points lie together in a $\epsilon_2/2$-region. Furthermore, we may suppose that all these chains $[C_i{}^1]$ contain the same number, say $2^{v_2} + 1$, of points which we denote by

$$X_i{}^1 = X^2{}_{i \cdot 2^{v_2}}, X^2{}_{i \cdot 2^{v_2}+1}, \cdots, X^2{}_{(i+1)2^{v^2}} = X^1{}_{i+1}.$$

Clearly the chains $[C_i{}^1]$ taken in order form an $\epsilon_2/2$-chain C^2 from a to b which contains all points of P_{n_2} and is such that any two successive points lie together in an $\epsilon_2/2$-region.

Let us continue in this manner. In general, for each k, we choose an integer $n_k \geq n_{k-1}$ such that every point of K lies in an $\epsilon_k/2$-region with some point of P_{n_k}. For each i, $0 \leq i \leq 2^{v_1+v_2+\cdots+v_{k-1}}$, let $F_i{}^{k-1}$ be the set of all points of P_{n_k} which lie together with $X_i{}^{k-1}$ in an $\epsilon_{k-1}/2$-region. Then there exists an ϵ_{k-1}-region $M_i{}^{k-1}$ containing $X_i{}^{k-1} + F_i{}^{k-1} + X^{k-1}_{i+1}$. By the lemma,

M_i^{k-1} contains an $\epsilon_k/2$-chain C_i^{k-1} from X_i^{k-1} to X_{i+1}^{k-1} which contains all points of F_i^{k-1} and is such that any two successive points lie together in an $\epsilon_k/2$-region. We may suppose that all these chains $[C_i^{k-1}]$ contain the same number, say $2^{v_k}+1$, of points. Clearly, then, these chains $[C_i^{k-1}]$, taken in the order $C_1^{k-1}, C_2^{k-1}, \cdots, C_{2^{v_1+v_2+\cdots+v_{k-1}}}^{k-1}$ form an $\epsilon_k/2$-chain C^k of points from a to b which contains P_{n_k} and is such that any two successive points lie together in an $\epsilon_k/2$-region and which consists of exactly $2^{v_1+v_2+\cdots+v_k}+1$ points:

$$a = X_0^k, X_1^k, \cdots, X^k{}_{2^{v_1+v_2+\cdots+v_k}} = b.$$

Furthermore, in this notation we have always:

(i) $$X^k{}_{i\cdot 2^{v_k}} = X_i^{k-1}.$$

Now let D be the set of all dyadic rational numbers on the interval $(0,1)$ and let t be any number of D. Then for some k we can write

$$t = i/2^{v_1+v_2+\cdots+v_k}, \qquad (0 \leq i \leq 2^{v_1+v_2+\cdots+v_k}).$$

To this value of t we make correspond the point X_i^k and write $T(t) = X_i^k$. Thus we define the transformation T over D. It is a consequence of (i) that T is single valued on D. Furthermore T is uniformly continuous on D, as is seen by identically the same argument as given for a similar purpose in the author's paper "Concerning continuous images of the interval." *

Finally, let $H = T(D)$. Then since $H = \sum_1^\infty C^n$ and for each n, $C^n \supset P_{n_n} = \sum_1^n p_i$, it follows that $H \supset \sum_1^\infty p_i = P$; and since P is dense in K, therefore H is dense in K. This completes the proof.

4. *Consequences of* § 3. (a). Suppose that the set M as above treated is a subset of some complete space. In this case it follows that since D is compact, so also is H. And since T is uniformly continuous on D, it is well known that the definition of T can be extended to the limit points of D in one and only one way so that the extended transformation is single valued and continuous on $D = (0,1)$. Thus $\bar{H}$ is the continuous image of an interval, and therefore it is a compact locally connected continuum. Obviously $\bar{H}$ contains K. Hence we have shown that *if the connected and locally connected set M is a subset of some complete space, then every self-*

* This Journal, Vol. 53 (1931), p. 673. Much similarity will be noted between the entire argument in the present paper and that in the paper here referred to. The latter in turn is very similar to an argument employed by Hahn in *Wiener Berichte*, Vol. 123 (1914), p. 2436.

compact subset K of M lies in a compact locally connected continuum which is a subset of $\bar{M}$.

(b). Suppose M itself is compact. Then we can take $K = M$ and thus we have $\bar{H} = M$. Then by the reasoning given under (a), we see that M is the continuous image of an interval: Thus we obtain the Hahn-Mazurkiewicz * theorem as a corollary to our proposition in § 3.

(c). Suppose M itself is complete. Then, treating M as the space, we have $\bar{H} \subset \bar{M} = M$. Thus by (a) we have that *any self-compact subset K of a connected and locally connected complete space M is contained in a compact locally connected continuum which is a subset of M.*

Thus, using the well known theorem that any compact locally connected continuum is arcwise connected (i. e. any two of its points are the extremities of a simple arc lying in it), we obtain the stronger proposition of Moore-Menger † that any connected and locally connected complete space is arcwise connected. Indeed, using (a), we obtain an even stronger result which may be stated as follows: *If M is any connected and locally connected subset of a complete space, then every two points of M can be joined by a simple continuous arc lying wholly in $\bar{M}$.*

It is to be noted, however, that this stronger result is a consequence also of some results of Menger's obtained in his article just referred to.

(d). By (a) we know that any complete space which contains a non-degenerate connected and locally connected set must contain compact locally connected continua and hence must contain simple continuous arcs. Thus in particular, a compact continuum which contains no arc ‡ *can contain no non-degenerate connected and locally connected set.* Since a continuum which is hereditarily locally connected, i. e., a continuum every subcontinuum of which is locally connected, has § the stronger property that *each of its connected subsets is locally connected,* we remark that the arcless continua and the hereditarily locally connected continua are diametric opposites not only as regards their subcontinua but also with respect to their non-degenerate connected subsets.

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* See Hahn, *loc. cit.*, and Mazurkiewicz, *Fundamenta Mathematicae*, Vol. 1 (1920), p. 166.

† See R. L. Moore, *Bulletin of the American Mathematical Society*, abstract, Vol. 33 (1927), p. 141; and K. Menger, *Monatshefte für Mathematik und Physik*, Vol. 36 (1930), p. 210.

‡ For examples, see Knaster, *Fundamenta Mathematicae*, Vol. 3 (1922), p. 247, and Whyburn, this Journal, Vol. 52 (1930), p. 319.

§ See R. L. Wilder, *Proceedings of the National Academy of Sciences*, Vol. 15 (1929), p. 616.

ON THE ANALYTICAL EXTENSION OF FUNCTIONS DEFINED BY FACTORIAL SERIES.*

By HOWARD K. HUGHES.

1. *Introduction.* In this paper we shall be concerned with the two kinds of infinite series known, respectively, as "factorial series of the first kind" and "factorial series of the second kind." A factorial series of the first kind is a series of the form

$$\sum_{n=0}^{\infty} a_n n!/z(z+1)\cdots(z+n), \tag{1}$$

while a factorial series of the second kind is one of the form

$$a_0 + \sum_{n+1}^{\infty} a_n(z-1)(z-2)\cdots(z-n)/n! = \sum_{n=0}^{\infty} a_n \binom{z-1}{n}. \tag{2}$$

In each case z represents a variable, real or complex, and the coefficients $a_0, a_1, a_2, \cdots$ are independent of z.

Series of form (1) are of fundamental importance in the study of the solutions of difference equations, and play a rôle which is analogous to that played by power series in expressing the solutions of differential equations.

On the other hand, series (2) is of less value in function theory since the representation of a given function $f(z)$ in such a series is sometimes possible in more than one way.† However series (2) frequently enters in problems in interpolation, it being seen that the coefficients a_n are equal to the successive differences of the function $f(z)$ at the point $z = 1$.

Factorial series apparently made their first appearance in the work of Newton and Stirling. They were used by Schlömilch about the middle of the nineteenth century in connection with studies concerning definite integrals. Nevertheless, it is not until recent years that a critical study has been made of the series in question particularly as to their convergence and analytic properties. This has been accomplished in large measure by Bendixson, Nielsen, Landau and Nörlund. A list of papers relative to such investigations will be found in the Bibliography. Still more recently, H. Bohr has discovered some results concerning the summability of series (1). Nörlund has investigated further the possibility of developing a given function $f(z)$ in

* Presented to the American Mathematical Society at Chicago, April 4, 1931.

† See, for example, Nörlund (5), pp. 224-226. (Bibliography at end of article.)

factorial series of both kinds, and has also discovered some noteworthy results on the analytical extension of functions defined by such series. He has employed extensively factorial series of the first kind in determining the solutions of linear difference equations. Moreover, certain generalized forms of series (1) and (2) have been studied by Pincherle, Carmichael, Fort, and others.

The object of this paper is to obtain new results pertaining to the analytical extension of a function such as is defined by series (1) or (2). As has been shown by Nielsen, the region of convergence of a factorial series is, in general, a half-plane, lying to the right of a vertical line in the z complex plane, this line being called the *boundary line of convergence.* Our problem is to extend analytically into regions to the left of the boundary line of convergence the function defined by such a series, the series itself being regarded as given.

Chapter I is expository, and sets forth some of the fundamental facts concerning factorial series as discovered by Nielsen, Landau, and others. We include also a brief discussion of the results already noted by Nörlund on the analytical extension of such series. Chapter II is devoted to results on analytical extension which are believed to be new, these results being embodied in three theorems. As to the methods employed, the proof of Theorem I rests upon the properties of a certain definite integral, while the proofs of Theorems II and III rest upon the calculus of residues. For this reason the chapter has been divided into two parts. Theorem I, including its proof, has been embodied in Part I, while Theorems II and III have been embodied in Part II. As a corollary to Theorem III, certain results pertaining to the asymptotic development of a function $W(z)$ defined by series (2) have been obtained. Several examples have been introduced to illustrate the results.

CHAPTER I.

Fundamental Theorems in the Theory of Factorial Series.

2. The purpose of this chapter is twofold. In the first place we shall present in concise form several theorems, now classical, concerning the convergence of the two kinds of series mentioned in the Introduction, namely,

$$\sum_{n=0}^{\infty} a_n n!/z(z+1)\cdots(z+n) \tag{1}$$

and

$$\sum_{n=0}^{\infty} a_n \binom{z-1}{n}. \tag{2}$$

These theorems are fundamental in what follows. In the second place we shall indicate briefly the results which have been obtained by Nörlund on the analytical extension of functions such as are defined by series (1) and (2).

The domain of convergence of factorial series has been studied by Nielsen, Pincherle, Landau, and others. Moreover, Nörlund has deduced a noteworthy result concerning the region of uniform convergence. The more important results pertaining to both convergence and uniform convergence, and the consequences thereof, and those concerning analytical extension will now be briefly stated, the proofs, however, being omitted.

I * (a). If series (1) converges for $z = z_o$, then it converges for all values of z for which $R(z) > R(z_o)$ † except at the points $z = 0, -1, -2, \cdots$.

(b). If series (2) converges for $z = z_o$, where z_o is not a positive integer, then it converges for all values of z for which $R(z) > R(z_0)$.

In case the series converges neither everywhere nor nowhere, it follows from the foregoing theorem that the region of convergence of series (1) or (2) is a half-plane, bounded on the left by a line of the form $R(z) = \lambda$, the number λ being called the *abscissa of convergence*. The series converges everywhere to the right of the line $R(z) = \lambda$, and diverges everywhere to the left of the same line. On the line the series may either converge or diverge. In the case of series (1), the points $z = 0, -1, -2, \cdots$, should any such points lie to the right of the boundary line of convergence, are excluded from the domain of convergence, since at these points the terms of the series become infinite. Series (2) may also be said to converge for positive integral values of z, should any such points lie to the left of the line $R(z) = \lambda$, since at these points the series reduces to a polynomial. Theorems will be stated presently showing how λ may actually be determined for any given series of form (1) or (2).

Landau has proved the following theorem concerning the uniform convergence of series (1) and (2):

II. Each of the two series (1) and (2) converges uniformly in any region R which, together with its boundary, lies interior to the half-plane of convergence of the series, provided in the case of series (1) that R does not include any of the points $z = 0, -1, -2, \cdots$.

Moreover Nörlund has shown that if series (1) or (2) converges at $z = z_0$ (in case of series (2), $z_0 \neq$ positive integer), then it converges uniformly in any sector δ whose vertex is at z_0, and which is characterized by the inequalities

* See Nielsen (2), pp. 416-429; (8), p. 124, 235; also Bendixson (1), pp. 15-20.

† Whenever the symbol $R(z)$ occurs, it is understood to mean the *real part of* z.

$$0 \leqq |z - z_0| \leqq G, \qquad -\pi/2 + \epsilon \leqq \arg(z - z_0) \leqq \pi/2 - \epsilon,$$

where G is any positive number, and ϵ is an arbitrarily small positive quantity. In case of series (1), the neighborhoods of the points $z = 0, -1, -2, \cdots$, should any such points lie in δ, are excepted.

As an immediate consequence of II, we have

III. Each of the two series (1) and (2) represents an analytic function in its half-plane of convergence, the neighborhoods of the points $z = 0, -1, -2, \cdots$ being excepted in the case of series (1).

We next note the manner in which the actual value of λ pertaining to any given series (1) or (2) may be determined. For this purpose, let us set

$$\alpha = \overline{\lim_{n\to\infty}} \log \left| \sum_0^n a_n \right| / \log n, \qquad \beta = \overline{\lim_{n\to\infty}} \log \left| \sum_0^{n-1} (-1)^n a_n \right| / \log n,$$

$$\alpha' = \overline{\lim_{n\to\infty}} \log \left| \sum_{n+1}^{\infty} a_n \right| / \log n, \qquad \beta' = \overline{\lim_{n\to\infty}} \log \left| \sum_n^{\infty} (-1)^n a_n \right| / \log n.$$

Then we have the following theorem: *

IV. The abscissa λ of convergence of series (1) is given by α or α' according as the series $\sum_{n=0}^{\infty} a_n$ is divergent or convergent; and the abscissa λ of convergence of series (2) is given by β or β' according as the series itself is divergent or convergent for $z = 0$.

As limiting cases we may have $\lambda = -\infty$ or $\lambda = +\infty$, corresponding to which cases the series converges or diverges respectively for all values of z.

3. We shall now mention briefly some results obtained by Nörlund on the analytical extension of functions such as are defined by series (1) and (2). We have the following four theorems: †

V. Any function $\Omega(z)$ which can be developed in a series of form (1) can also be developed in the form

$$\Omega(z) = \sum_{n=0}^{\infty} c_n n!/(z+\omega)(z+2\omega)\cdots(z+n\omega) \tag{3}$$

where ω is any given constant. Let λ_ω denote the corresponding abscissa of convergence. Moreover let ω_1 be any fixed value of ω such that $\omega_1 > 1$, and let ω' be any variable value such that $\omega' > \omega_1$. Then $\lim_{\omega'\to\infty} \lambda_{\omega'}$ exists, and is such that

$$\lambda_{\omega'} \leqq \lambda_{\omega_1} \leqq \lambda.$$

* Landau (1), p. 156, 195; Nörlund (5), p. 223, 257.

† Nörlund (1); (2), pp. 341-379; (5), pp. 233-240, and 262-266.

As an immediate consequence of V, we have the following:

VI. Series (3) when the (variable) quantity ω is real and greater than unity, furnishes the analytical extension of the function $\Omega(z)$ defined by (1) throughout the strip

$$\lambda_\omega < R(z) \leqq \lambda$$

except in the neighborhoods of the points $z = 0, -1, -2$, etc.

VII. Any function $W(z)$ which can be developed in a series of form (2) can also be developed in the form

$$W(z) = \sum_{n=0}^{\infty} c_n(z-\omega)(z-2\omega)\cdots(z-n\omega)/n! \tag{4}$$

where ω is any given constant. Let λ_ω represent the corresponding abscissa of convergence. Moreover let ω_1 be any fixed value of ω, and ω' any variable value such that $0 < \omega' < \omega_1 < 1$, then $\lim_{\omega' \to 0} \lambda_{\omega'}$ exists and is such that

$$\lambda_{\omega'} \leqq \lambda_{\omega 1} \leqq \lambda.$$

Hence we have, corresponding to VI above, the following:

VIII. Series (4), when the (variable) quantity ω is real and such that $0 < \omega < 1$, furnishes the analytical extension of the function $W(z)$ defined by (2) throughout the strip

$$\lambda\omega < R(z) \leqq \lambda.$$

CHAPTER II.

Analytical Extension of Functions Defined by Factorial Series.

I. *Analytical Extension By Use of a Loop Integral.*

4. As has been stated in the Introduction, it is the purpose of this paper to derive new results concerning the analytical extension of functions defined by factorial series. In the results of Nörlund, mentioned in the preceding chapter, the given series is transformed into another factorial series whose line of convergence, in general, lies to the left of the line of convergence, $R(z) = \lambda$, of the given series. Thus, these results furnish a means of obtaining the analytical extension of the function defined by the given series throughout a vertical strip of determinate width lying to the left of the line $R(z) = \lambda$. It is our purpose to obtain expressions which will furnish the analytical extension of such a function over the whole finite z plane, when the coefficients of the given series satisfy certain conditions.

For our present purpose, let us now write series (1) of the Introduction in the form

$$(1) \qquad \sum_{n=0}^{\infty} g(n)/z(z+1)\cdots(z+n)$$

where it is understood that z is a variable, and that the coefficients $g(n)$ depend only on n.

We start with a result shown incidentally by Ford in his paper, "On the Behavior of Integral Functions in Distant Portions of the Plane," * namely that the (entire) function $f(t,z)$ defined by the special series

$$f(t,z) = \sum_{n=0}^{\infty} t^n/\Gamma(z+n+1)$$

may be written in the following form for any value (real or complex) of t, and for any value of z such that $R(z) > 0$:

$$(2) \qquad f(t,z) = \frac{e^t t^{-z}}{\Gamma(z)} \int_0^t e^{-t} t^{z-1} dt.$$

If we restrict ourselves to values of t such that $R(t) > 0$, we may evidently write (2) in the form

$$(3) \qquad f(t,z) = \frac{e^t t^{-z}}{\Gamma(z)} \left\{ \int_0^{\infty} e^{-t} t^{z-1} dt - \int_t^{\infty} e^{-t} t^{z-1} dt \right\},$$

where it is understood that in the first integral the integration takes place along the positive axis of reals from $t = 0$ to $t = \infty$, while in the second integral it takes place in the direction of the positive real axis from $t = t$ to $t = \infty$. We have now only to recall that

$$\int_0^{\infty} e^{-t} t^{z-1} dt = \Gamma(z)$$

in order to write (3) in the form

$$(4) \qquad f(t,z) = e^t t^{-z} \left\{ 1 - \frac{1}{\Gamma(z)} \int_1^{\infty} e^{-t} t^{z-1} dt \right\}$$

But the right hand member here appearing is defined not only when $R(z) > 0$, but is seen to have a meaning whatever be the value of z. Moreover in the neighborhood of any finite point z it is seen to be analytic.

If in particular we take t equal to unity in (4), and multiply both members by $\Gamma(z)$, the product $\Gamma(z) \cdot f(1,z)$ occurring on the left is seen to be the function $G(z)$ defined by the special factorial series

* *Bulletin of American Mathematical Society*, Vol. 34 (1928), p. 91. We here change Ford's notation slightly in order to accommodate his result more aptly to our subsequent discussion.

$$\sum_{n=0}^{\infty} 1/z(z+1)\cdots(z+n).$$

Thus, for the function $G(z)$ defined by this special series, we may write

$$G(z) = e\{\Gamma(z) - \int_1^{\infty} e^{-t}t^{z-1}dt\}.$$

Since the right member of this equation preserves a meaning and is an analytic function of z throughout any region of the z plane which does not include any of the points $z = 0, -1, -2, \cdots$, it furnishes the value of $G(z)$ throughout any such region.

We shall now show that this special result may be generalized so as to obtain an analogous one which will apply to factorial series of the first kind in general, at least under some restrictions on the coefficients. In fact, such a generalization is possible as follows:

THEOREM I. *Having given the factorial series*

$$\sum_{n=0}^{\infty} g(n)/z(z+1)\cdots(z+n) \tag{1}$$

let the following power series be formed:

$$g(0) + \frac{\Delta g(0)}{1!}(1-t) + \frac{\Delta^2 g(0)}{2!}(1-t)^2 + \cdots$$

where

$$\Delta g(0) = g(1) - g(0), \qquad \Delta^2 g(0) = g(2) - 2g(1) + g(0),$$

etc. Suppose that the function $h(t)$ defined by this series can be expanded about the origin in a series whose radius of convergence exceeds unity. Suppose further that $h(t)$ is analytic throughout an infinite strip of the t plane which shall include the positive real t axis together with the origin, and which may be of arbitrarily small width. Furthermore suppose that the function $h(t)$ is such that the improper integral

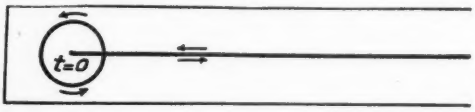
$$\int_0^{\infty} h(t)e^{-t}t^{z-1}dt$$

exists, at least for $R(z) > 0$. Then the function $\Omega(z)$ defined by (1) may be expressed for all values of z in the form*

$$\Omega(z) = \frac{e}{e^{2\pi i z} - 1}\int_L h(t)e^{-t}t^{z-1}dt - e\int_1^{\infty} h(t)e^{-t}t^{z-1}dt, \tag{5}$$

* For further comments on these conditions, see Article 5.

where the first integral is the loop integral obtained by an integration about the origin from the point at infinity along the upper side of the positive real t axis to the neighborhood of the origin, and returning to infinity along the lower side of the same axis, as indicated in the figure.



An immediate corollary of this theorem is as follows: If the given series (1) has a finite abscissa of convergence, then relation (5) furnishes the analytical extension of the function $\Omega(z)$ throughout all finite portions of the z plane which do not contain any of the points $z = 0, -1, -2, \cdots$.

In order to prove the theorem, let us start by considering the expression

$$F(t,z) = e^t \cdot t^{-z} \int_0^t h(t) e^{-t} t^{z-1} dt \tag{6}$$

where z has any value (real or complex) such that $R(z) > 0$, and where we shall regard $h(t)$ as any function which satisfies the conditions imposed in the hypothesis. Under such conditions the integrand in (6) can be expanded in powers of t, and the series can be integrated between the limits 0 and t provided $|t| \leqq 1$. It follows that we may write the function $F(t,z)$ in the form

$$F(t,z) = c_0 + c_1 t + c_2 t^2 + \cdots, \tag{7}$$

where the coefficients c_n are as follows:

$$c_0 = h(0)/z$$
$$c_1 = h(0)/z + [h'(0) - h(0)]/(z+1)$$
$$c_2 = h(0)/z + [h'(0) - h(0)]/1!(z+1) + [h''(0) - 2h'(0) + h(0)]/2!(z+2)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$c_n = \sum_{s=0}^{n} G(s)/(n-s)!\,(z+s),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

where

$$G(s) = \sum_{r=0}^{s} (-1)^r \binom{s}{r} h^{(s-r)}(0).$$

But if we write c_1 in the form

$$c_1 = A/z + B/z(z+1),$$

upon clearing of fractions and equating coefficients of like powers of z, thus determining the values of A and B, we find

$$c_1 = h'(0)/z + [h(0) - h'(0)]/z(z+1).$$

If we do the same for c_2, c_3, etc., the coefficients in (7) are found to take the following form:

$$\begin{aligned}
c_0 &= h(0)/z \\
c_1 &= h'(0)/z + [h(0) - h'(0)]/z(z+1) \\
c_2 &= h''(0)/2!\,z + [h'(0) - h''(0)]/1!\,z(z+1) \\
&\qquad + [h(0) - 2h'(0) + h''(0)]/z(z+1)(z+2) \qquad (8) \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
c_n &= \sum_{s=0}^{n} H(s)/(n-s)!\,z(z+1)\cdots(z+n),
\end{aligned}$$

where

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$H(s) = \sum_{r=0}^{s} (-1)^r \binom{s}{r} h^{(n+r-s)}(0).$$

For our present purpose, let us now set t in (7) equal to unity. The result is

$$F(1, z) = c_0 + c_1 + c_2 + \cdots$$

where the values of c_n are given by (8). We shall next collect the terms in $1/z$, $1/z(z+1)$, etc. The coefficient of $1/z$ is

$$h(0) + h'(0) + h''(0)/2! + \cdots,$$

or, simply $h(1)$. The coefficient of $1/z(z+1)$ is

$$\{h(0) + h'(0) + h''(0)/2! + \cdots\} - \{h'(0) + h''(0) + h'''(0)/2! + \cdots\}$$

or $h(1) - h'(1)$. Having continued indefinitely, we may now write

$$\begin{aligned}
F(1, z) &= h(1)/z + [h(1) - h'(1)]/z(z+1) \\
&\qquad + [h(1) - 2h'(1) + h''(1)]/z(z+1)(z+2) + \cdots \qquad (9) \\
&\quad \cdots + \sum_{s=0}^{n} (-1)^s \binom{n}{s} h^{(s)}(1)/z(z+1)\cdots(z+n) + \cdots,
\end{aligned}$$

where it is to be noted that the right member is a factorial series of the first kind. But by (6) we see that, by placing $t = 1$, the function defined by series (9) is given by the integral

$$F(1, z) = e \int_0^1 h(t) e^{-t} t^{z-1} dt \qquad (10)$$

provided that $R(z) > 0$.

This result having been established, let us now consider the series

$$\sum_{n=0}^{\infty} g(n)/z(z+1)\cdots(z+n) \qquad (1)$$

with the purpose of determining the function $h(t)$ so that the two series (1) and (9) shall coincide, thus causing the function $\Omega(z)$ defined by (1) to be given by the integral (10). In order that the two series shall coincide, it is necessary and sufficient that

$$\begin{aligned}
&g(0) = h(1),\\
&g(1) = h(1) - h'(1),\\
&g(2) = h(1) - 2h'(1) + h''(1),\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&g(n) = \sum_{n=0}^{\infty} (-1)^s \binom{n}{s} h^{(s)}(1),\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{aligned}$$

These equations are equivalent to the following:

$$\begin{aligned}
&g(0) = h(1),\\
&\Delta g(0) = -h'(1),\\
&\Delta^2 g(0) = h''(1),\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&\Delta^n g(0) = (-1)^n h^{(n)}(1),\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{aligned}$$

Hence we must have

$$h(t) = g(0) + \Delta g(0)\ (1-t) + \Delta^2 g(0)\,(1-t)^2/2! + \cdots.$$

If this function $h(t)$ satisfies the conditions indicated in the hypothesis of the theorem, we can now assert that the function $\Omega(z)$ defined by series (1) may be written in the form

$$\Omega(z) = e\int_0^1 h(t)e^{-t}t^{z-1}dt. \tag{11}$$

provided $R(z) > 0$. If the abscissa of convergence of (1) is greater than zero, then it follows that (11) furnishes the analytical extension of the function $\Omega(z)$ throughout that portion of the z plane lying between the boundary line of convergence and the axis of pure imaginaries.

In order to extend this result so as to obtain the analytical extension of the function $\Omega(z)$ over the whole finite z plane, let us now write the integral appearing in (11) in the form

$$\int_0^1 h(t)e^{-t}t^{z-1}dt = \int_0^\infty h(t)e^{-t}t^{z-1}dt - \int_1^\infty h(t)e^{-t}t^{z-1}dt. \tag{12}$$

By hypothesis, the first integral on the right exists when $R(z) > 0$. When regarded as a function of z this integral is seen to be analytic throughout

any portion of the half-plane $R(z) > 0$, while the second integral is seen at once to be analytic for all values of z. Inasmuch as the first integral may fail to preserve a meaning for $R(z) < 0$, we proceed to study the function $J(z)$ defined by this integral. In this connection let us consider the loop integral

$$l(z) = \int_L h(t)e^{-t}t^{z-1}dt, \tag{13}$$

the contour L extending from infinity along the upper side of the positive real t axis to the point $t = \epsilon$, (ϵ being an arbitrarily small positive number), thence around the circle of radius ϵ about the origin, and returning to infinity along the lower side of the same axis. It will be sufficient for the moment to consider z as real and positive. If we let $t = r(\cos\theta + i\sin\theta)$, then we have $t^{z-1} = \exp[(z-1)(\log r + i\theta)]$, $0 \leqq \theta \leqq 2\pi$. Along the upper side of the positive real t axis, the angle θ is 0, while along the lower side we have $\theta = 2\pi$. The contribution to $l(z)$ arising from integrating along the upper side of the axis from $t = \infty$ to $t = \epsilon$ is given by the integral

$$-\int_\epsilon^\infty h(t)e^{-t}t^{z-1}dt$$

while the contribution arising from integrating along the lower side from $t = \epsilon$ to $t = \infty$ is given by

$$e^{2\pi iz}\int_\epsilon^\infty h(t)e^{-t}t^{z-1}dt.$$

As ϵ approaches zero, the sum of these two contributions approaches as a limit the expression

$$(e^{2\pi iz} - 1)J(z)$$

where

$$J(z) = \int_0^\infty h(t)e^{-t}t^{z-1}dt.$$

Upon the circle, we have $t^{z-1} = (\epsilon e^{i\theta})^{z-1}$. Moreover the product $h(t)\cdot e^{-t}$, being analytic at the origin, remains less than a constant k in this neighborhood. Hence the contribution to $l(z)$ arising from integrating around the circle is in absolute value less than

$$k\epsilon^z \left| \int_0^\pi e^{iz\theta}d\theta \right| = 2\pi k\epsilon^z.$$

Therefore, as ϵ approaches zero, the contribution in question vanishes. Consequently, we have

$$J(z) = l(z)/(e^{2\pi iz} - 1). \tag{14}$$

Suppose now we let z take on complex values. The left member of (14) is an analytic function of z for $R(z) > 0$. But the right member preserves a meaning and is analytic for all values of z except $z = 0, -1, -2, \cdots$. Hence (14) must hold for all values of z except zero and negative integers. Taking account of (12), and (13), we have, finally,

$$\Omega(z) = e/(e^{2\pi i z} - 1) \int_L h(t) e^{-t} t^{z-1} dt - e \int_1^\infty h(t) \cdot e^{-t} t^{z-1} dt \tag{15}$$

Hence the proof of the theorem is complete.

5. *Remarks.* The following remarks relative to the foregoing theorem are now to be noted.

1′. We have required that the function $h(t)$ shall admit an expansion in powers of t which shall have a radius of convergence exceeding unity. If, however, the radius is unity, but the series can be integrated termwise from 0 to 1, then Theorem I still holds provided that the other conditions required of $h(t)$ are satisfied.

2′. We require $h(t)$ to be analytic throughout an infinite strip of arbitrarily small width which shall include the positive real t axis and the origin. This condition will evidently be satisfied by any entire function.

3′. We have required further that the function $h(t)$ be such that the integral

$$\int_0^\infty h(t) e^{-t} t^{z-1} dt$$

shall exist for $R(z) > 0$. If the condition stated in remark 2′ is satisfied, then this integral will evidently exist whenever the following statement can be made: Corresponding to any arbitrarily small positive number ϵ there exists a positive constant K_ϵ such that for all values of t sufficiently large, we may write

$$|h(t)| < K_\epsilon e^{t(1-\epsilon)}.$$

We note in this connection that all the conditions imposed on $h(t)$ are satisfied in the special case in which the general coefficient $g(n)$ is a polynomial, for then the function $h(t)$ is also a polynomial.

4′. Nielsen * has shown that if series (1) of the Introduction converges, then the function $\Omega(z)$ defined by that series may be expressed in the form

$$\Omega(z) = \int_0^1 \phi(t) t^{z-1} dt, \tag{16}$$

* (8), pp. 239-241.

where $\phi(t)$ is the function defined by the power series

$$a_0 + a_1(1-t) + a_2(1-t)^2 + \cdots.$$

Moreover Pincherle * has shown that an integral of the type here appearing exists and is an analytic function of z throughout a half-plane $R(z) > \sigma$, where the abscissa σ may be greater than, equal to, or less than the abscissa λ of convergence of the corresponding factorial series.

That the general result obtained in Theorem I gives the particular result of Nielsen when $R(z) > \max(0, \sigma)$ may be shown as follows: We have defined $h(t)$ in the form

$$h(t) = g(0) + \Delta g(0)\,(1-t) + \Delta^2 g(0)(1-t)^2/2! + \cdots.$$

Furthermore we may write

$$e^{-t} = (1/e\,[1 + (1-t) + (1-t)^2/2! + \cdots].$$

Upon multiplying these two series, we obtain

$$\begin{aligned}(17)\quad h(t)\cdot e^{-t} = (1/e)\,[&g(0) + \{g(0) + \Delta g(0)\}(1-t) \\ &+ \{\Delta^2 g(0)/2! + \Delta g(0) + g(0)/2!\}(1-t)^2 + \cdots \\ &+ [(\Delta+1)^n g(0)/n!\,](1-t)^n + \cdots].\end{aligned}$$

But

$$\begin{aligned}&g(0) + \Delta g(0) = g(1) = a_1 \\ &\Delta^2 g(0)/2! + \Delta g'(0) + g(0)/2! = g(2)/2! = a_2,\end{aligned}$$

and in general

$$(\Delta+1)^n g(0) = g(n)/n! = a_n.$$

Therefore we may write

$$h(t)e^{-t} = (1/e)\,[a_0 + a_1(1-t) + a_2(1-t)^2 + \cdots],$$

or

$$h(t)e^{-t} = 1/e \cdot \phi(t).$$

Consequently, for $R(z) > \max(0, \sigma)$, we have

$$(18)\qquad e\int_0^1 h(t)e^{-t}t^{z-1}dt = \int_0^1 \phi(t)\,t^{z-1}dt.$$

Now when $R(z) > 0$, the quantity

$$\frac{1}{e^{2\pi i z} - 1}\int_L h(t)e^{-t}t^{z-1}dt$$

* See Ref. (3).

occurring in the right member of (15) reduces to the integral

$$\int_0^\infty h(t)e^{-t}t^{z-1}dt$$

and therefore for $R(z) > 0$ the whole right member reduces at once to the left member of (18). The equivalence of the two results for $R(z) > \max(0, \sigma)$ is thus established. It appears, therefore, that the result of Theorem I goes further than Nielsen's result since the former furnishes the value of the function $\Omega(z)$ throughout the whole finite z plane while the latter furnishes the value of the same function only, in general, for values of z in a certain half-plane.

Examples. 1. Let $\Omega(z)$ be defined by the series

$$\sum_{n=0}^{\infty} n^2/z(z+1)\cdots(z+n).$$

Then we have $h(t) = t^2 - 3t + 2$. Hence we have

$$\Omega(z) = \int_0^1 e^{1-t}(t^2 - 3t + 2)\cdot t^{z-1}dt.$$

This integral fails to have a meaning for $R(z) < 0$, since the integrand becomes infinite at the lower limit. But by Theorem I we have, for any finite z,

$$\Omega(z) = \frac{e}{e^{2\pi iz} - 1}\int_L (t^2 - 3t + 2)e^{-t}t^{z-1}dt - e\int_1^\infty (t^2 - 3t + 2)e^{-t}t^{z-1}dt.$$

2. Consider the series

$$\sum_{n=0}^{\infty} 2^n/z(z+1)\cdots(z+n).$$

Then we have

$$\begin{aligned} h(t) &= 1 + (1-t) + (1/2!)(1-t)^2 + \cdots, \\ &= e^{1-t}. \end{aligned}$$

and therefore we may express $\Omega(z)$ in the form

$$\Omega(z) = e^2\int_0^1 e^{-2t}t^{z-1}dt.$$

However, this integral has no meaning for $R(z) < 0$, but applying Theorem I we have for any finite value of z,

$$\Omega(z) = \frac{e^2}{e^{2\pi iz} - 1}\int_L e^{-2t}t^{z-1}dt - e^2\int_1^\infty e^{-2t}t^{z-1}dt.$$

II. *Analytical Extension by the Calculus of Residues.*

6. In this portion of the present chapter we shall employ a method which differs essentially from that used above. In fact we shall employ the calculus of residues to obtain the analytical extension of functions defined by both kinds of factorial series under the assumption that the coefficients alternate in sign, and satisfy certain further conditions.

The calculus of residues has been used by Barnes, Ford, and others in finding the analytical extension of functions defined by various types of power series. As consequences of the results obtained, certain further results pertaining to the asymptotic developments of such functions have been established. The question naturally presents itself as to whether the method mentioned can be used also to obtain analogous results for factorial series. We shall show, in fact, that some such results are possible.

Our investigations are based upon the following fundamental theorem which is a direct consequence of elementary considerations in the theory of functions of a complex variable: *

"If $P(w)$ and $Q(w)$ are any two functions of the complex variable w, both of which are single valued and analytic in a region R of the w complex plane, and of which the latter vanishes at the points $w = \lambda_1, \lambda_2, \cdots \lambda_n$, which are zeroes of the first order, and if C_n denote any contour lying in R and inclosing the points $w = \lambda_1, \lambda_2, \cdots \lambda_n$, then we have

$$\frac{1}{2\pi i}\int_{C_n}\frac{P(w)}{Q(w)}\,dw = \sum_{t=1}^{n}\frac{P(\lambda_t)}{Q'(\lambda_t)}, \tag{19}$$

where the integration is performed in the positive sense."

Our first theorem is as follows:

THEOREM II. *Given any factorial series of the form*

$$\sum_{n=0}^{\infty}\frac{(-1)^n g(n)}{z(z+1)\cdots(z+n)} \tag{20}$$

whose abscissa λ of convergence is finite. If the function $g(n)$ occurring in the coefficient of this series is such that, when considered as a function $g(w)$ of the complex variable $w = x + iy$, (a) it is single valued and analytic throughout all portions of the w plane lying to the right of (or upon) the vertical line $w = -\frac{1}{2} + iy$, except for a finite number of poles situated at the points $w = \lambda_1, \lambda_2, \cdots \lambda_n$, ($\lambda_t \neq$ integer), and (b) to any arbitrarily small

* See, for example, Ford, "Studies on Divergent Series and Summability," *Michigan Science Series*, Vol. II, 1916, page 184.

positive number ϵ there corresponds a positive constant K_ϵ such that for all values of $x \geqq -\frac{1}{2}$ and for all positive values of y sufficiently large we may write

$$\left| \frac{g(x \pm iy)}{g(x)} \right| < K_\epsilon e^{y(\pi/2-\epsilon)},$$

then the function $\Omega(z)$ defined by series (20) when $R(z) > \lambda$ may be extended analytically throughout the whole finite z plane, except in neighborhoods of the points $z = 0, -1, -2, \cdots$, and throughout this region will be defined by the equation

$$\Omega(z) = \frac{\Gamma(z)}{2} \int_{-\infty}^{+\infty} \frac{g(-\frac{1}{2}+iy)\,dy}{\Gamma(z+\frac{1}{2}+iy)\cosh \pi y} - \sum_{t=1}^{n} r_t \tag{21}$$

where r_t represents the residue of the function

$$\pi\Gamma(z)g(w)/\Gamma(z+w+1)\sin \pi w \tag{22}$$

at the point $w = \lambda_t$.

The proof of this theorem is analogous to that of the second part of the general theorem proved by Ford in his paper, "On the Behavior of Integral Functions in Distant Portions of the Plane," already mentioned in the first part of this chapter.

In order to prove the theorem, let us write series (20) in the form

$$\sum_{n=0}^{\infty} (-1)^n \Gamma(z)g(n)/\Gamma(z+n+1). \tag{23}$$

Furthermore, for simplicity, let us assume first that the function $g(w)$ has no poles to the right of the line $w = -\frac{1}{2} + iy$. Moreover, let us regard z as having any fixed value which we shall take as real, positive, greater than λ, and not an integer. The theorem then results, as we shall show, from integrating the function (22) about the rectangular contour C_n formed in the w plane by the lines

$$w = -1/2 + iy, \qquad w = 1/2 + 2n + iy, \qquad w = x \pm ij,$$

where n is any positive integer, and j is an arbitrarily large positive number.

Taking

$$P(w) = \frac{\pi\Gamma(z)g(w)}{\Gamma(z+w+1)}, \qquad Q(w) = \sin \pi w,$$

and applying (19), we arrive at the relation

$$\Gamma(z) \sum_{n=0}^{2n} \frac{(-1)^n g(n)}{\Gamma(z+n+1)} = \frac{\Gamma(z)}{2i} \int_{C_n} \frac{g(w)\,dw}{\Gamma(z+w+1)\sin \pi w}. \tag{24}$$

We now proceed to study the integral here appearing in further detail.

First, along that side of C_n upon which $w = x + ij$, we have $dw = dx$, $\sin \pi w = \sin \pi j \, (\sin \pi x \cosh \pi j + i \cos \pi x)$. Hence if we denote by I the contribution to the contour integral in (24) arising from integrating along this side, we have

$$I = \frac{\Gamma(z)}{2i \sin h\pi j} \int_{2n+\frac{1}{2}}^{-\frac{1}{2}} \frac{g(x+ij)\,dx}{\Gamma(z+x+1+ij)\,(\sin \pi x \cosh \pi j + i \cos \pi x)}.$$

Now it is a well known property of the Gamma function that if α and β are real, then we may write *

$$(25) \qquad |\,\Gamma(\alpha+i\beta)\,| = (2\pi)^{\frac{1}{2}} \,|\,\alpha+i\beta\,|^{\alpha-\frac{1}{2}} e^{-\alpha-\beta \tan^{-1}\beta/\alpha} \cdot (1+\delta)$$

where δ approaches zero as α or β becomes infinite. Moreover as $j \to +\infty$, $\sinh \pi j$ becomes infinite like $e^{\pi j}$, and when we take account of condition (b) of the hypothesis, it appears that the absolute value of the integrand in I vanishes to as high an order as that of $e^{-\epsilon j} \,|\, z+x+1+ij \,|^{z+x+\frac{1}{2}}$ as $j \to +\infty$. Hence we have at once $\lim_{j\to\infty} I = 0$.

Similarly, the contribution arising from integrating along that side of C_n upon which $w = x - ij$ will be seen to approach zero as $j \to +\infty$. For as the expression in question differs from I above only in that j is replaced by $-j$, and as $j \to +\infty$ the function $\sinh(-\pi j)$ becomes infinite like $e^{\pi j}$, the absolute value of the integrand vanishes like $e^{-\epsilon j} \,|\, z+x+1-ij \,|^{z+x+\frac{1}{2}}$ as $j \to +\infty$.

Next, along that side of C_n along which $w = 2n + \frac{1}{2} + iy$, we have $dw = idy$, $\sin \pi w = \cosh \pi y$. If J represents the contribution to the contour integral (24) from the integration along this side, then we have

$$(26) \qquad J = \frac{\Gamma(z)}{2} \int_{-\infty}^{+\infty} \frac{g(\frac{1}{2}+2n+iy)\,dy}{\Gamma(z+2n+3/2+iy)\cosh \pi y}.$$

As $y \to +\infty$, $\cosh \pi y$ becomes infinite like $e^{\pi y}$, while as $y \to -\infty$ the same function becomes infinite like $e^{-\pi y}$. Upon noting relation (25) and condition (b) of the hypothesis, we see that as $y \to +\infty$, the absolute value of the integrand vanishes to as high an order as that of $e^{-\epsilon y} \,|\, z+2n+\frac{3}{2}+iy \,|^{z+2n+1}$, while as $y \to -\infty$, it vanishes to as high an order as that of the quantity $e^{\epsilon y} \,|\, z+2n+\frac{3}{2}-iy \,|^{z+2n+1}$. It follows, therefore, that the improper integral appearing in (26) exists. Moreover we may now show $\lim_{n\to\infty} J = 0$. To do this, let us write J in the form

* Godefroy, *La Fonction Gamma*, Paris, 1901, page 45.

$$J = \tfrac{1}{2} \cdot \frac{\Gamma(z) \cdot g(2n)}{\Gamma(z+2n+1)} \int_{-\infty}^{+\infty} \frac{g(2n+\tfrac{1}{2}+iy)}{g(2n)} \cdot \frac{\Gamma(z+2n+1)}{\Gamma(z+2n+\tfrac{3}{2}+iy)} \cdot \frac{dy}{\cosh \pi y}.$$

Since the given series is assumed convergent for $R(z) > \lambda$, we have at once

$$\lim_{n\to\infty} \frac{\Gamma(z)\, g(2n)}{\Gamma(z+2n+1)} = 0.$$

Furthermore, from (25), we may write

$$\left| \frac{\Gamma(z+2n+1)}{\Gamma(z+2n+\tfrac{3}{2}+iy)} \right| < \frac{|\, z+2n+\tfrac{3}{2}\,|^{z+2n+1}}{|\, z+2n+\tfrac{3}{2}+iy\,|^{z+2n+1}}\, e^{-y \tan^{-1}[y/(2n+z+3/2)]} < e^{\pi/2|\, y\,|}.$$

Finally, taking account of condition (b) of the hypothesis, we have $\lim_{n\to\infty} J = 0$.

Along the fourth side of C_n, we have $w = -\tfrac{1}{2} + iy$, $dw = i dy$, $\sin \pi w = -\cosh \pi y$, and the integration takes place from $+\infty$ to $-\infty$. If we now take into consideration the contribution to the contour integral arising from integrating along this side, the fundamental relation (24) takes the following form (n having been allowed to become infinite as indicated above):

$$\Omega(z) = \frac{\Gamma(z)}{2} \int_{-\infty}^{+\infty} \frac{g(-\tfrac{1}{2}+iy)\, dy}{\Gamma(z+\tfrac{1}{2}+iy)\cosh \pi y}. \tag{27}$$

Up to this point we have confined z to values which are real, positive, and greater than λ. Suppose now that z be allowed to take on any value, real or complex. The left member of (27) is an analytic function of z when $R(z) > \lambda$, except in neighborhoods of the points $z = 0, -1, -2, \cdots$. We may show, however, that the right member represents an analytic function of z throughout any finite portion T of the z plane which does not — any of the points $z = 0, -1, -2, \cdots$. To establish this fact it suffices to show that the improper integral involved converges uniformly throughout any such region. For this purpose, consider the integral

$$\int_l^{\infty} \frac{g(-\tfrac{1}{2}+iy)\, dy}{\Gamma(z+\tfrac{1}{2}+iy)\cosh \pi y}. \tag{28}$$

From (25) and condition (b) of the hypothesis, it appears that the absolute value of the integrand for a sufficiently large y is less than

$$e^{-\epsilon y}\, |\, z_1 + \tfrac{1}{2} + iy\,|^{z_1}$$

where z_1 denotes the maximum absolute value of z in the proposed region T. But we may write

$$e^{-\epsilon y} \mid z_1 + \tfrac{1}{2} + iy \mid^{z_1} = e^{-(\epsilon/2)y} \cdot e^{-(\epsilon/2)y} \mid z_1 + \tfrac{1}{2} + iy \mid^{z_1} < Ke^{-(\epsilon/2)y}$$

where K is a constant. Hence we have

$$\left| \int_l^\infty \frac{g(-\tfrac{1}{2} + iy)\,dy}{\Gamma(z + \tfrac{1}{2} + iy)\cosh \pi y} \right| < K \int_l^\infty e^{-(\epsilon/2)y} dy = (2/\epsilon) K e^{-\epsilon l/2}$$

provided l is a sufficiently large quantity independent of z. Consequently the integral (28) can be made less in absolute value than any arbitrarily small positive number η by proper choice of the lower limit l. In the same way, we can show that

$$\left| \int_{-\infty}^{-l} \frac{g(-\tfrac{1}{2} + iy)\,dy}{\Gamma(z + \tfrac{1}{2} + iy)\cosh \pi y} \right| < \eta$$

provided l is sufficiently large. Thus the uniform convergence of the improper integral in (27) is established. The relation must therefore hold for all values of z except zero and negative integers.

The proof for the special case in which the function $g(w)$ has no poles in the half-plane $R(w) \geqq -\tfrac{1}{2}$ is thus complete. That the theorem is true when we allow $g(w)$ to have poles as indicated in condition (a) of the hypothesis follows when we note that relation (24) continues to hold (n being sufficiently large) provided we add to its left member the expression

$$\sum_{t=0}^{n} r_t$$

as it is defined in the statement of the theorem.

7. *Remarks.* 1′. The line $w = -\tfrac{1}{2} + iy$ taken as the left side of the contour C_n can be replaced by any line $w = k + iy$ where $-1 < k < 0$ provided that $g(w)$ is analytic when $R(w) \geqq k$, except for a finite number of poles as indicated in condition (a).

2′. Should the given series converge throughout the whole z plane, then the function $\Omega(z)$ is expressed at all points z by (21).

3′. Suppose condition (a) is not satisfied, but the function $g(w)$ has a finite number of branch points situated at the points $w = \lambda_1, \lambda_2, \cdots \lambda_n$. Then (21) continues to hold provided one adds to the left member the sum of the loop integrals at these points. The p-th of these is described as follows: Draw the straight line extending from the point λ_p to infinity in the direction of the positive axis of imaginaries, and let this line be regarded as a cut in the w plane. Now let the loop consist of the two lines drawn on either side of and near the cut, the ends of these lines in the neighborhood

of the point λ_p being joined by a circular arc of small radius drawn about this point. The integration is to be performed in the positive sense.

Examples.

1. Consider the function $\Omega(z)$ defined by the series

$$\Omega(z) = \sum_{n=0}^{\infty} (-1)^n n!/z(z+1)\cdots(z+n).$$

This series converges when $R(z) > 0$, and may be shown to be represented by the integral

$$\int_0^1 [t^{z-1}/(2-t)]\,dt$$

which in turn preserves a meaning for $R(z) > 0$. Since the function $n!$ or $\Gamma(n+1)$ satisfies both conditions (a) and (b) of the hypothesis of Theorem II, then by that theorem we may write

$$\Omega(z) = \frac{\Gamma(z)}{2} \int_{-\infty}^{+\infty} \frac{\Gamma(\tfrac{1}{2}+iy)\,dy}{\Gamma(z+\tfrac{1}{2}+iy)\cosh \pi y},$$

which holds for all values of z except $0, -1, -2, \cdots$.

2. Let $g(n) = h(n)\,n!$, where the function $h(w)$ satisfies condition (a), while corresponding to any arbitrarily small positive number ϵ there exists a positive constant K_ϵ such that

$$|\,h(x \pm iy)/h(x)\,| < K_\epsilon e^{y(\pi-\epsilon)}, \qquad x \geqq -\tfrac{1}{2}.$$

then the function $g(w)$ satisfies both conditions (a) and (b), and, assuming that the given series converges, we have

$$\Omega(z) = \frac{\Gamma(z)}{2} \int_{-\infty}^{+\infty} \frac{h(-\tfrac{1}{2}+iy)\Gamma(\tfrac{1}{2}+iy)\,dy}{\Gamma(z+\tfrac{1}{2}+iy)\cosh \pi y},$$

this result holding for all finite values of z different from zero and negative integers.

8. So far our results have been concerned with the factorial series of the first kind. We shall now turn our attention to the second kind of factorial series. For this purpose, let us write series (2) of the Introduction in the form

$$g(0) + \sum_{n=1}^{\infty} g(n)\,(z-1)\,(z-2)\cdots(z-n)$$

where, as above, z denotes a variable, and where the general coefficient $g(n)$ depends only on n. Then we have the following theorem which is analogous to Theorem II:

THEOREM III. *Given any factorial series of the form*

$$(29) \qquad g(0) + \sum_{n=1}^{\infty} (-1)^n g(n)(z-1)(z-2)\cdots(z-n),$$

whose abscissa of convergence is finite. If the function $g(n)$ occurring in the general coefficient of this series is such that when considered as a function $g(w)$ of the complex variable $w = x + iy$ (a) it is single-valued and analytic throughout all portions of the half-plane $R(w) \geqq -\frac{1}{2}$, except for a finite number of poles situated at the points $w = \lambda_1, \lambda_2, \cdots \lambda_n$, none of which are integers, and (b) to any arbitrarily small positive number ϵ there corresponds a positive constant K_ϵ such that for $x \geqq -\frac{1}{2}$ and for all positive values of y sufficiently large we may write

$$\left| \frac{g(x \pm iy)}{g(x)} \right| < K_\epsilon e^{y(\pi/2-\epsilon)}$$

then the function $W(z)$ defined by series (29) may be extended analytically throughout all finite portions of the z plane except in the neighborhoods of the points $z = 0, -1, -2, \cdots$, and throughout such regions will be defined by the equation

$$W(z) = \frac{\Gamma(z)}{2} \int_{-\infty}^{+\infty} \frac{g(-\frac{1}{2} + iy)\,dy}{\Gamma(z + \frac{1}{2} - iy)\cosh \pi y} - \sum_{t=1}^{n} r_t$$

where r_t represents the residue of the function

$$\frac{\pi g(w) \cdot \Gamma(z)}{\Gamma(z-w)\sin \pi w}$$

at the point $w = \lambda_t$.

In fact the given series may be written in the form

$$\sum_{n=0}^{\infty} (-1)^n g(n)\Gamma(z)/\Gamma(z-n)$$

and the proof of the theorem is analogous to the proof of Theorem I above. It will therefore be left to the reader.

9. If the function $g(w)$ is single valued and analytic throughout the half-plane $R(w) \geqq -q - \frac{1}{2}$, where q is any positive integer, then by taking the line $w = -q - \frac{1}{2} + iy$ as the left side of the contour C_n used in the proofs of the last two theorems, we may show that the following equation holds:

$$(30) \qquad W(z) = \sum_{n=-1}^{-q} \frac{(-1)^n g(n)\Gamma(z)}{\Gamma(z-n)} + \frac{(-1)^q \Gamma(z)}{2} \int_{-\infty}^{+\infty} \frac{g(-q-\frac{1}{2}+iy)\,dy}{\Gamma(z+q+\frac{1}{2}-iy)\cosh \pi y}.$$

If $g(w)$ has a finite number of poles situated at the points $w = \lambda_1, \lambda_2, \cdots \lambda_n$, none of which are integers, then we must add to the left member of (30) the sum of the residues of the function

$$\pi g(w)\Gamma(z)/\Gamma(z-w)\sin \pi w$$

at the points λ_t.

As a consequence of (30), we have the following

COROLLARY. If the coefficient $g(n)$ occurring in the general coefficient of series (29), when considered as a function of the complex variable $w = x + iy$, is such that (a) it is single-valued and analytic throughout the finite w plane, except for a finite number of poles $\lambda_1, \lambda_2, \cdots \lambda_n$, none of which is an integer, and (b) to any arbitrarily small positive number ϵ there corresponds a positive constant K_ϵ such that for all values of x and for all values of y sufficiently large we may write

$$\left| \frac{g(x \pm iy)}{g(x)} \right| < K_\epsilon e^{y(\pi/2-\epsilon)},$$

then the function $W(z)$ defined by (29) will be such that for all values of z lying in any sector (vertex at $z = 0$) which does not include the negative real axis, we may write

$$W(z) \sim -\sum_{t=1}^{n} r_t + \frac{g(-1)}{z} - \frac{g(-2)}{z(z+1)} + \frac{g(-3)}{z(z+1)(z+2)} - \cdots.$$

In fact the expression

$$\Gamma(z) \int_{-\infty}^{+\infty} \frac{g(-q-\tfrac{1}{2}+iy)\,dy}{\Gamma(z+q+\tfrac{1}{2}-iy)\cosh \pi y} \tag{31}$$

may be written in the form

$$\frac{\Gamma(z)}{\Gamma(z+q+\tfrac{1}{2})} \int_{-\infty}^{+\infty} \frac{g(-q-\tfrac{1}{2}+iy)\,\Gamma(z+q+\tfrac{1}{2})\,dy}{\Gamma(z+q+\tfrac{1}{2}-iy)\cosh \pi y}$$

or

$$\frac{\Gamma(z)}{\Gamma(z+q+\tfrac{1}{2})} \int_{-\infty}^{+\infty} \frac{g(-q-\tfrac{1}{2}+iy)B(z+q-\tfrac{1}{2}, 1-iy)\,dy}{\Gamma(z-iy)\cosh \pi y}$$

where B is the customary symbol for the Beta function. Now we have from the definition of the Beta function

$$|B(z+q-\tfrac{1}{2}, 1-iy)| < \left| \int_0^1 x^{z+q-3/2}(1-x)^{-iy}dx \right|$$
$$< \left| \int_0^1 x^{z+q-3/2}dx \right| = 1/|z+q-\tfrac{1}{2}|.$$

Recalling that

$$(z+q-\tfrac{1}{2})\Gamma(z+q-\tfrac{1}{2}) = \Gamma(z+q+\tfrac{1}{2}),$$

we see that the absolute value of the remainder (31) is less than

$$\left| \frac{\Gamma(z)}{\Gamma(z+q+\tfrac{1}{2})} \right| \cdot \int_{-\infty}^{+\infty} \frac{|\, g(-q-\tfrac{1}{2}+iy)\,|\, dy}{|\,\Gamma(1-iy)\,|\cosh \pi y}.$$

By virtue of (25) and condition (b) of the hypothesis, the integral here appearing exists. Furthermore, by (25), the expression

$$\left| \frac{\Gamma(z)}{\Gamma(z+q+\tfrac{1}{2})} \right|$$

vanishes to as high an order as the $(q-\tfrac{1}{2})-th$ as $|\, z\,| \to \infty$, provided that z does not go to infinity along the negative real axis. Hence the corollary follows.

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ON SEPARATION, COMPARISON AND OSCILLATION THEOREMS FOR SELF-ADJOINT SYSTEMS OF LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS.†

By G. A. Bliss and I. J. Schoenberg.

In order to indicate the character of the theorems in this paper let us consider for a moment the differential equation

$$y'' + p(x)y' + q(x)y = 0, \tag{1}$$

in which the coefficients $p(x)$ and $q(x)$ are real, single-valued, and continuous functions of x for $x_1 \leqq x \leqq x_2$. The complete set of zeros on x_1x_2 of a solution $y = y(x)$ of (1) which vanishes at $x = \xi$ but is not identically zero on x_1x_2 has no limit point on x_1x_2 and is therefore finite, and the point ξ defines uniquely the whole set of zeros, say $\xi_{-l}, \cdots, \xi_{-1}, \xi, \xi_1, \cdots, \xi_k$. Sturm ‡ proved the following

Separation Theorem. *The sets of zeros on x_1x_2 of two different solutions of* (1) *coincide or else separate each other, i. e. between two adjacent points of one set there is one and only one point of the second set.*

Let us consider further two differential equations of a more special type

$$y'' + A(x)y = 0, \tag{2}$$

$$y'' + A^*(x)y = 0, \tag{3}$$

in which $A(x)$ and $A^*(x)$ are continuous on x_1x_2 and such that

$$A(x) > A^*(x) \text{ for } x_1 \leqq x \leqq x_2. \tag{4}$$

A consequence of these assumptions is Sturm's §

Comparison Theorem. *The zeros of a solution of* (2) *are closer together than the zeros of a solution of* (3). *More precisely: let $\xi^*_{-l}, \cdots, \xi^*_{-1}, \xi, \xi^*_1, \cdots, \xi^*_k$, be a complete set of zeros of a solution of* (3) *on x_1x_2, then ξ*

† Presented to the Society June 13, 1931.

‡ C. Sturm, "Mémoire sur les équations différentielles du second ordre," *Journal de Mathématiques pures et appliquées*, Vol. 1 (1836), pp. 106-186.

§ See C. Sturm, *loc. cit.*, p. 125, where a more general theorem is given.

defines a set of zeros for a solution of (2) *on* x_1x_2 *which has at least* $k+l+1$ *consecutive points* $\xi_{-l}, \cdots, \xi_{-1}, \xi, \xi_1, \cdots, \xi_k$, *and these are such that*

$$\xi_\alpha < \xi^*_\alpha \quad (\alpha = 1, 2, \cdots, k), \quad \xi_{-\beta} > \xi^*_{-\beta} (\beta = 1, 2, \cdots, l).$$

Finally let

$$y'' + A(x, \lambda) y = 0, \tag{5}$$

be a differential equation for which $A(x, \lambda)$ has the following properties:

1. $A(x, \lambda)$ is supposed to be a continuous function of (x, λ) for $x_1 \leqq x \leqq x_2$, and $\lambda \geqq \lambda_0$.

2. For values λ', λ'' satisfying $\lambda_0 \leqq \lambda' < \lambda''$ we suppose

$$A(x, \lambda') < A(x, \lambda''), \text{ for } x_1 \leqq x \leqq x_2.$$

3. For every $M > 0$ arbitrarily large, there exists a number $\lambda(M)$, such that $A(x, \lambda) > M$, for $x_1 \leqq x \leqq x_2$ and $\lambda > \lambda(M)$.

4. The interval x_1x_2 contains no two consecutive zeros of a solution of (5), taken for $\lambda = \lambda_0$.

The comparison theorem makes it possible to prove under these assumptions the following

OSCILLATION THEOREM.† *There is a uniquely determined sequence of numbers* $(\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m < \cdots)$ *with*

$$\lim_{m \to \infty} \lambda_m = \infty,$$

such that

$$y'' + A(x, \lambda_m) y = 0, \qquad (m = 1, 2, \cdots),$$

has a solution vanishing at $x = x_1$ *and* $x = x_2$, *and which vanishes precisely* $m - 1$ *times between* x_1 *and* x_2.

The assumptions 1, 2, 3, and 4, are always satisfied for $A(x, \lambda) = \lambda A(x)$, $A(x)$ being continuous and positive on x_1x_2, and $\lambda_0 = 0$. In this case, the sequence λ_m has the following further property: *The series* $\sum_{m=1}^{\infty} \lambda_m^{-\alpha}$ *converges for* $\alpha > \frac{1}{2}$ *and diverges for* $\alpha \leqq \frac{1}{2}$.

This paper gives a generalization of these three theorems for a self-adjoint system of linear second order differential equations. The theorems

† This theorem goes back to Sturm. For the statement given here see L. Bieberbach, *Differentialgleichungen*, 2nd edition, Berlin (1926), pp. 155-156.

concern what we call a conjugate system of points with respect to such a system of equations. Our separation theorem includes results of J. Radon † on conjugate points in the calculus of variations. The comparison theorem, which is related to a comparison theorem of Morse ‡, will be established quite independently with the help only of methods which are classical in the calculus of variations. This theorem makes it possible to extend to this more general situation the well known continuity proof for the Sturm-Liouville oscillation theorem.

1. The Generalized Separation Theorem.

We consider the quadratic form

$$2\Omega(x, y, y') = P_{ik}(x)y_iy_k + 2Q_{ik}(x)y_iy_k' + R_{ik}(x)y_i'y_k', \tag{1.1}$$

its coefficients P_{ik}, Q_{ik}, R_{ik}, being functions of x of class C' † on x_1x_2, with $P_{ik} = P_{ki}$, $R_{ik} = R_{ki}$, and

$$R_{ik}(x)v_iv_k > 0 \text{ for } v_iv_i > 0, \quad x_1 \leqq x \leqq x_2. \tag{1.2}$$

We are to deal with the system

$$J_i(y) \equiv (d/dx)\Omega_{y_i'} - \Omega_{y_i} = 0, \qquad (i = 1, 2, \cdots, n), \tag{1.3}$$

which is the most general self-adjoint system of second order linear differential equations. Condition (1.2) implies $|R_{ik}| \neq 0$ on x_1x_2 and therefore (1.3) can be solved for the second derivatives y_i'' and put into the normal form on the interval $x_1 \leqq x \leqq x_2$. Its solutions $y_i = y_i(x)$, which we are using later, are supposed to be of class C'' § and not identically zero on x_1x_2.

One readily verifies the following identities

$$\begin{gathered} y_i\Omega_{y_i} + y_i'\Omega_{y_i'} = 2\Omega, \; y_i\Omega_{z_i} + y_i'\Omega_{z_i'} = z_i\Omega_{y_i} + z_i'\Omega_{y_i'}, \\ y_iJ_i(z) - z_iJ_i(y) = (d/dx)\,(y_i\Omega_{z_i'} - z_i\Omega_{y_i'}). \end{gathered} \tag{1.4}$$

For two solutions $y_i = y_i(x)$, $z_i = z_i(x)$ of (1.3), one therefore has $y_i\Omega_{z_i'} - z_i\Omega_{y_i'} = \text{const}$. These two solutions are said to be conjugate if

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* J. Radon, "Zum Problem von Lagrange," *Abhandlungen aus dem Mathematischen Seminar Hamburg*, Vol. 6, pp. 298-299. The methods used in our § 1 may be applied without great change to the discussion of conjugate points for the Lagrange problem.

‡ M. Morse, "A Generalization of the Sturm Separation and Comparison Theorems in n-Space," *Mathematische Annalen*, Vol. 103, pp. 52-69. For geometric applications of this theorem see I. J. Schoenberg, "Some Applications of the Calculus of Variations to Riemannian Geometry (to appear in the *Annals of Mathematics*).

§ A function is of class C' if it is continuous and has a continuous first derivative. The function is of class C'' if also the second derivative is continuous.

$y_i\Omega_{z_i'} - z_i\Omega_{y_i'} = 0$. We shall call *a conjugate system of solutions* of (1.3), a system of n solutions $u_{1k}, u_{2k}, \ldots, u_{nk}$, $(k = 1, 2, \cdots, n)$, which are conjugate to each other and linearly independent on x_1x_2.

The solution $y_i = y_i(x)$ is said to vanish at $x = \xi$ if $y_i(\xi) = 0$, $(i = 1, 2, \cdots, n)$. We shall prove the following

LEMMA 1.1.† *If u_{ik} is a conjugate system of solutions with $U \equiv |u_{ik}| \neq 0$ on x_1x_2, then no solution $y_i = y_i(x)$ of (1.3) can vanish twice on the interval x_1x_2.*

Let $y_i = y_i(x)$ be some solution of (1.3). Because of $U \neq 0$, $y_i = u_{ik}a_k$ defines uniquely a set of functions $a_k = a_k(x)$ on x_1x_2. We put

$$y_i = u_i = u_{ik}a_k, \qquad v_i = u_{ik}a_k'. \tag{1.5}$$

Primes attached to expressions involving u_i or v_i will always indicate derivatives of those expressions with respect to x calculated as if the coefficients a_k, a_k' were independent of x. One readily verifies, then, the relations

$$\begin{gathered}(\Omega_{u_i'})' = \Omega_{u_i},\ (\Omega_{v_i'})' = \Omega_{v_i},\ u_i\Omega_{v_i'} - v_i\Omega_{u_i'} = 0,\\ (d/dx)\Omega_{u_i'} = (\Omega_{u_i'})' + \Omega_{v_i'} = \Omega_{u_i} + \Omega_{v_i'}\end{gathered} \tag{1.6}$$

in which it is understood that the differentiation indicated by d/dx takes account of the fact that the coefficients a_k are functions of x. With the same convention concerning differentiation

$$y_i' = u_{ik}'a_k + u_{ik}a_k' = u_i' + v_i. \tag{1.7}$$

With the help of Taylor's formula, the first equation (1.4), and the equations (1.6) and (1.7), one readily verifies the further relations

$$\begin{aligned}2\Omega(x, y, y') &= 2\Omega(x, u, u' + v)\\ &= 2\Omega(x, u, u') + 2v_i\Omega_{u_i'} + R_{ik}v_iv_k\\ &= u_i\Omega_{u_i} + u_i'\Omega_{u_i'} + 2v_i\Omega_{u_i'} + R_{ik}v_iv_k\\ &= u_i(\Omega_{u_i} + \Omega_{v_i'}) + (u_i' + v_i)\Omega_{u_i'} + R_{ik}v_iv_k\end{aligned}$$

$$2\Omega(x, y, y') = (d/dx)(y_i\Omega_{u_i'}) + R_{ik}v_iv_k. \tag{1.8}$$

Hence with the help of the first equation (1.4)

$$y_i(\Omega_{y_i} - (d/dx)\,\Omega_{y_i'}) + (d/dx)y_i(\Omega_{y_i'} - \Omega_{u_i'}) = R_{ik}v_iv_k. \tag{1.9}$$

This relation and condition (1.2) show that $y_i = y_i(x)$ can vanish at $x = \xi$,

† Compare G. A. Bliss, "The Problem of Lagrange in the Calculus of Variations," *American Journal of Mathematics*, Vol. 52 (1930), § 32, where the same fact is proved for the general Lagrange problem.

$x = \xi'$, only if $v_i = u_{ik}a_k' \equiv 0$ for $\xi \leqq x \leqq \xi'$; therefore $a_k' \equiv 0$ and hence the a_k are constant on $\xi\xi'$ in which case $y_i = u_{ik}a_k$ can not vanish at all on x_1x_2 since $U \neq 0$ on x_1x_2.

LEMMA 1.2. *A conjugate system of solutions* (u_{ik}) *can not have* $U = |u_{ik}| \equiv 0$ *on any subinterval of* x_1x_2.

If $U \equiv 0$ on some subinterval of x_1x_2, then one can certainly choose a subinterval $\xi\xi'$ sufficiently small so that the same minor of U of maximum order shall be $\neq 0$ on $\xi\xi'$. This permits us to select functions $a_k = a_k(x)$ of class C' on $\xi\xi'$ with $y_i = u_{ik}a_k \equiv 0$ and not all $a_k(\xi) = 0$. Formula (1.9) gives $v_i \equiv 0$ on $\xi\xi'$ and from (1.7) one concludes that $u_{ik}'a_k \equiv 0$ on $\xi\xi'$. The set of functions $u_i = u_{ik}c_k$, with $c_k = a_k(\xi)$, is therefore a solution of (1.3) which has the properties $u_i(\xi) = 0$, $u_i'(\xi) = 0$, and which hence vanishes identically on x_1x_2. But $u_{ik}c_k \equiv 0$ on x_1x_2 contradicts the definition of a conjugate system of solutions.

LEMMA 1.3. *There exists a conjugate system of solutions whose elements all vanish at an arbitrarily selected point* $x = \xi$. *There also exists a conjugate system of solutions whose determinant* $U^* = 1$ *at an arbitrarily selected point* $x = \xi$.

The proof follows readily with the help of the canonical variables

$$z_i = \Omega_{y_i'} = Q_{ki}y_k + R_{ik}y_k'. \tag{1.10}$$

Solving with respect to y_k' we get

$$y_k' = G_k(x, y, z)$$

which is linear and homogeneous in y_i and z_i. In terms of the canonical variables x, y_i, z_i, (1.3) takes the normal form

$$\begin{aligned} dy_i/dx &= G_i(x, y, z), \\ dz_i/dx &= P_{ik}y_k + Q_{ik}G_k(x, y, z). \end{aligned} \tag{1.11}$$

A solution of (1.3) or (1.11) is uniquely determined by giving the values of y_i and z_i at some initial point $x = \xi$. The set of n solutions $y_i = u_{ik}$, $z_i = v_{ik}$, $(k = 1, 2, \cdots, n)$, defined by the initial values

$$u_{ik}(\xi) = 0, \quad v_{ik}(\xi) = \delta_{ik}, \tag{1.12}$$

is readily shown to be a conjugate system of solutions satisfying the requirements for the first conjugate system of our lemma. The initial values

$$u_{ik}(\xi) = \delta_{ik}, \quad v_{ik}(\xi) = 0, \tag{1.13}$$

similarly lead at once to the second conjugate system required by the lemma. We will designate further on by $U(\xi, x)$ and $U^*(\xi, x)$ the determinants of these special systems defined by the initial conditions (1.12) and (1.13) respectively.

LEMMA 1.4. *Let $y_i = y_i(x)$ be a solution of (1.3) with $y_i(\xi) = 0$, and let (u_{ik}) be the first conjugate system of solutions defined for $x = \xi$ in Lemma 1.3. The solution $y_i(x)$ is linearly expressible in the form $y_i = u_{ik}c_k$ with constant coefficients c_k.*

The solution $y_i = y_i(x)$ defines a set $z_i = z_i(x)$ by means of (1.10). Let (u_{ik}) be the conjugate system of solutions defined by (1.12). We determine the constants c_k from $z_i(\xi) = v_{ik}(\xi)c_k$. The solution $y_i - u_{ik}c_k$, $z_i - v_{ik}c_k$, of (1.11) has initial values all zero at $x = \xi$ and hence vanishes identically on x_1x_2. This proves our lemma.

Definition. The points ξ and ξ' on x_1x_2 are said to be associated if there is a solution $y_i = y_i(x)$ of (1.3) with not all $y_i(x) \equiv 0$ on $\xi\xi'$ and having $y_i(\xi) = 0$, $y_i(\xi') = 0$, $(i = 1, 2, \cdots, n)$.

THEOREM 1.1. *If on the interval x_1x_2 a point ξ has a following (preceding) associated point, then it also has a first following (preceding) associated point ξ_1. This point ξ_1, which we shall call the following (preceding) conjugate point to ξ, is given by the first root of the determinant $U(\xi, x) = 0$ following (preceding) the value $x = \xi$.*

It is sufficient to make the proof for the conjugate point following the point $x = \xi$. Let ξ' be a following associated point of ξ. We set up the conjugate system of solutions defined by (1.12). One readily proves that ξ' is a root of $U(\xi, x) = 0$, because by definition there is a solution $y_i = y_i(x)$ vanishing at ξ and ξ'. By lemma 1.4 there are constants c_k with $y_i = u_{ik}c_k$, and $y_i(\xi') = u_{ik}(\xi')c_k = 0$ gives $U(\xi, \xi') = 0$.

Let conversely ξ' be a point following ξ, with $U(\xi, \xi') = 0$. One determines constants c_k by means of the system $u_{ik}(\xi')c_k = 0$, and $y_i(x) = u_{ik}(x)c_k$ has the property $y_i(\xi) = y_i(\xi') = 0$, which shows that ξ and ξ' are associated.

All we have to show is the existence of a first root ξ_1 of $U(\xi, x) = 0$, following the point $x = \xi$. Because $U(\xi, x)$ is a continuous function of x, it is sufficient to show that $x = \xi$ is an isolated root of $U(\xi, x) = 0$. This last point follows from lemma 1.1 and lemma 1.3 and our arguments thus far. The conjugate system with $U^*(\xi, \xi) = 1$ (Lemma 1.3) defines some

interval containing ξ where $U^*(\xi, x) \neq 0$. By lemma 1.1 there are in this interval no two associated points and hence also no point associated to ξ. Hence $U(\xi, x) = 0$ has no root in this interval except $x = \xi$. This proves our theorem 1.1.

THEOREM 1.2. *If the point ξ_1 is conjugate to the point ξ, then also ξ is conjugate to ξ_1.*

It is sufficient to make the proof for $\xi < \xi_1$. Let ξ_2 be the conjugate point to ξ_1 preceding ξ_1. Certainly $\xi \leqq \xi_2$ (because ξ and ξ_1 are associated) and all we have to show is that actually $\xi = \xi_2$. Let us assume $\xi < \xi_2 < \xi_1$. By theorem 1.1 ξ_1 is the first zero of $U(\xi, x) = 0$ following ξ, and ξ_2 is the first zero of $U(\xi_1, x) = 0$ preceding ξ_1. Let ξ' be any point with $\xi < \xi' < \xi_2$. The determinant $U(\xi', x)$ vanishes at $x = \xi'$ but can not vanish for $\xi' < x < \xi_1$, because the interval $\xi\xi_1$ can not contain two associated points (theorem 1.1, lemma 1.1). Furthermore $U(\xi', \xi_1) \neq 0$ would imply a contradiction with lemma 1.1, ξ_2 and ξ_1 being associated while $U(\xi', x)$ would be $\neq 0$ on $\xi_2\xi_1$. One therefore has $U(\xi', \xi_1) = 0$ for every ξ' with $\xi < \xi' < \xi_2$. Hence ξ_1 is associated with every such ξ' and it results (theorem 1.1) that $U(\xi_1, x) \equiv 0$ on $\xi \leqq x \leqq \xi_2$, which result contradicts lemma 1.2. Hence $\xi = \xi_2$ and ξ_2 is actually conjugate to ξ_1.

This theorem makes possible the following

Definition. A set of points $\xi_1 < \xi_2 < \cdots < \xi_k$ on the interval x_1x_2, such that each is conjugate to the adjacent points of the set while ξ_1 has no preceding and ξ_k no following conjugate point on x_1x_2, is said to form a conjugate system of points.

THEOREM 1.3. *Every point ξ on x_1x_2 has associated with it a unique conjugate system which is completely defined by any one of its points.*

Every point ξ defines a conjugate $\xi_1 > \xi$ or else none $> \xi$ on x_1x_2 (theorem 1.1). The point ξ_1 defines a conjugate $\xi_2 > \xi_1$ or else none $> \xi_1$ (theorem 1.1). In this way one defines the points of the conjugate system following ξ. There is only a finite number, because the set $\xi, \xi_1, \xi_2, \cdots$ so determined can have no limit point, not even x_2. This last statement follows readily from lemma 1.1 and lemma 1.3. Similarly there is a finite set $\xi_{-1}, \xi_{-2}, \cdots, \xi_{-l}$ of points preceding ξ.

With the help of lemma 1.1, one readily proves the

SEPARATION THEOREM 1.4. *Two systems of conjugate points coincide or else separate each other, i. e., between two adjacent points of one system there is one and only one point of the second system.*

2. THE GENERALIZED COMPARISON THEOREM.

It will be the purpose of this section to generalize the Sturm comparison theorem, described in the introduction, to systems of the type (1.3) by means of the notion of a conjugate system of points as worked out in § 1.

Let $\Omega(x, y, y')$ be defined by (1.1), hypothesis (1.2) being retained. Besides the system

$$J_i(y) = (d/dx)\Omega_{y_i'} - \Omega_{y_i} = 0, \tag{2.1}$$

we set up the integral

$$I(y) = \int_{x_1}^{x_2} 2\Omega(x, y, y')\,dx. \tag{2.2}$$

LEMMA 2.1.† *If there is on x_1x_2 no point conjugate to x_1 with respect to the system* (2.1), *then we always have*

$$I(y) > 0, \tag{2.3}$$

for every set of functions $y_i = y_i(x)$ of class D' ‡ on x_1x_2, with $y_i(x_1) = y_i(x_2) = 0$, and which does not vanish identically on x_1x_2.

In order to prove this lemma we shall prove first the existence of a conjugate system of solutions of (2.1) with its determinant $\neq 0$ on x_1x_2. To show this, we extend the definition of P_{ik}, Q_{ik}, R_{ik}, on a slight extension $x_1 - \delta \leqq x \leqq x_1$ of x_1x_2. We do this in a way which preserves the continuity properties of the coefficients of $\Omega(x, y, y')$ and also (1.2). We suppose this δ sufficiently small in order to have $U^*(x_1, x) \neq 0$ on $x_1 - \delta \leqq x \leqq x_1 + \delta$, $U^*(x_1, x)$, with $U^*(x_1, x_1) = 1$, being the determinant of the second conjugate system of solutions introduced in lemma 1.3. The first determinant of lemma 1.3, $U(x_1, x)$, is a continuous function of the initial point x_1 and therefore it follows from $U(x_1, x) \neq 0$ on $x_1 + \delta \leqq x \leqq x_2$, that also $U(x_0, x) \neq 0$ for $x_1 + \delta \leqq x \leqq x_2$ and some x_0 on the interval $x_1 - \delta \cdots x_1$ and sufficiently close to x_1. From Lemma 1.1 and our assumption on $U^*(x_1, x)$, one gets also $U(x_0, x) \neq 0$ for $x_1 \leqq x \leqq x_1 + \delta$ and x_0 as described above. Hence $U(x_0, x) \neq 0$ on $x_1 \leqq x \leqq x_2$.

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† Compare J. Hadamard, *Calcul des variations* I, Paris, 1910, § 288.

‡ A function is of class D' on x_1x_2, if it is continuous, and if the interval x_1x_2 can be divided into a finite number of subintervals, such that in each subinterval its derivative is continuous.

Let $y_i = y_i(x)$ be a set of admissible functions. With our special conjugate system just defined and (1.5), formula (1.8) holds. One notices that $\Omega_{u_i'} = Q_{ki}u_k + R_{ik}u_k'$, with $u_k = u_{kj}a_j$ and $u_k' = u'_{kj}a_j$, is continuous, even at the possible corners of $y_i = y_i(x)$. Therefore integrating (1.8) we get †

$$I(y) = \int_{x_1}^{x_2} 2\Omega(x, y, y')\,dx = \int_{x_1}^{x_2} R_{ik}v_iv_k\,dx. \tag{2.4}$$

The interval x_1x_2 can be subdivided in a finite set of subintervals on each of which the v_i are continuous and not all identically zero. Our last formula (2.4) and (1.2) imply (2.3).

LEMMA 2.2. *If the points x_1 and x_2 are conjugate with respect to* (2.1), *then always*

$$I(y) \geqq 0, \tag{2.5}$$

for every set $y_i = y_i(x)$ of class D' on x_1x_2, with $y_i(x_1) = y_i(x_2) = 0$.

Let $y_i = y_i(x)$ be a set of class D' on x_1x_2, with $y_i(x_1) = y_i(x_2) = 0$, and defining an arc C_{12} in (x, y_i)-space. Let 3 and 4 be the points in (x, y_i)-space whose coördinates are $(x_3, y_i(x_3))$ and $(x_4, 0)$ respectively $(x_1 < x_3 < x_4 < x_2)$. Lemma 2.1 applied to the interval x_1x_4 shows that $I(C_{13} + L_{34} + L_{42}) = I(C_{13} + L_{34}) \geqq 0$, L_{34} and L_{42} being segments of straight lines in (x, y_i)-space. Keeping 3 fixed and moving 4 towards the point 2, we get at the limit $I(C_{13} + L_{32}) \geqq 0$. Certainly $I(C_{13} + L_{32})$ has the limit $I(C_{12}) = I(y)$, if 3 tends to the point 2. The last inequality insures therefore (2.5) and our lemma is proved.

We shall prove now the following

COMPARISON THEOREM 2.1. *Besides the system*

$$J_i(y) \equiv (d/dx)\Omega_{y_i'} - \Omega_{y_i} = 0, \tag{2.6}$$

we consider a second system of the same kind

$$J^*{}_i(y) \equiv (d/dx)\Omega^*{}_{y_i'} - \Omega^*{}_{y_i} = 0, \tag{2.7}$$

$2\Omega^*(x, y, y') = P^*{}_{ik}y_iy_k + 2Q^*{}_{ik}y_iy_k' + R^*{}_{ik}y_i'y_k'$ *having like Ω, its coefficients of class C' on x_1x_2.*

† For another method of deriving this so called Clebsch transformation of the second variation see G. A. Bliss, "The Transformation of Clebsch in the Calculus of Variations," *Proceedings of the International Mathematical Congress*, Toronto, 1924, Vol. 1, pp. 589-603.

We suppose

$$\Omega(x, u, v) \leqq \Omega^*(x, u, v), \tag{2.8}$$

for $x_1 \leqq x \leqq x_2$, *and arbitrary values of* u_i *and* v_i. *The inequalities* (2.8) *and* (1.2) *imply*

$$R^*_{ik}(x)v_iv_k > 0 \text{ for } v_iv_i > 0,\ x_1 \leqq x \leqq x_2. \tag{2.9}$$

Then a conjugate system of points for (2.6) *has its points closer together than a conjugate system of points for* (2.7). *More precisely: Suppose that*

$$\xi^*_{-l}, \cdots, \xi^*_{-1},\ \xi,\ \xi^*_1, \cdots, \xi^*_k,$$

is the conjugate system for (2.7) *defined by the point* ξ. *The point* ξ *will then define for* (2.6) *a conjugate system which has at least* $k + l + 1$ *consecutive points*

$$\xi_{-l}, \cdots, \xi_{-1}, \xi, \xi_1, \cdots, \xi_k,$$

with

$$\xi_\alpha \leqq \xi^*_\alpha\ (\alpha = 1, 2, \cdots, k), \text{ and } \xi_{-\beta} \geqq \xi^*_{-\beta}\ (\beta = 1, 2, \cdots, l). \tag{2.10}$$

RESTRICTED COMPARISON THEOREM 2.2. *We replace the assumption* (2.8) *by the stronger one:*

$$\Omega(x, u, v) < \Omega^*(x, u, v), \tag{2.11}$$

for $x_1 \leqq x \leqq x_2$, $u_iu_i > 0$, *and arbitrary values* v_i. *The statement as in theorem* 2.1 *then holds after strengthening the inequalities* (2.10) *to*

$$\xi_\alpha < \xi^*_\alpha (\alpha = 1, 2, \cdots, k), \text{ and } \xi_{-\beta} > \xi^*_{-\beta}\ (\beta = 1, 2, \cdots, l). \tag{2.12}$$

Proof of theorem 2.1: We shall prove this theorem by induction and we start by proving the existence of ξ_1 with $\xi_1 \leqq \xi^*_1$. The points ξ and ξ^*_1 being conjugate and therefore also associated for (2.7), there is a solution of (2.7) $y_i = u_i(x)$ of class C' on $\xi\xi^*_1$, with $u_i(\xi) = u_i(\xi^*_1) = 0$ and not $\equiv 0$ on $\xi\xi^*_1$. An integration by parts gives

$$\begin{aligned} I^*(u) &= \int_\xi^{\xi^*_1} 2\Omega^*(x, u, u')\,dx = \int_\xi^{\xi^*_1} (u_i\Omega^*_{u_i} + u_i'\Omega^*_{u_i'})\,dx \\ &= \int_\xi^{\xi^*_1} u_i(\Omega^*_{u_i} - (d/dx)\Omega^*_{u_i'})\,dx = 0. \end{aligned} \tag{2.13}$$

From (2.8) and (2.13) we get

$$I(u) = \int_\xi^{\xi^*_1} 2\Omega(x, u, u')\,dx \leqq \int_\xi^{\xi^*_1} 2\Omega^*(x, u, u')\,dx = I^*(u) = 0.$$

This last result and lemma 2.1 show that there is a point ξ_1 conjugate to ξ with respect to (2.6), with $\xi_1 \leqq \xi^*_1$.

Suppose now that we know already the existence of $\xi_1, \xi_2, \cdots, \xi_{m-1}$, with $\xi_\alpha \leqq \xi^*_\alpha$, for $\alpha = 1, 2, \cdots, m-1$, $(m \leqq k)$. We want to show that there is a ξ_m with $\xi_m \leqq \xi^*_m$. The same argument as above applies again. The points ξ^*_{m-1} and ξ^*_m are conjugate with respect to (2.7) and $\xi_{m-1} \leqq \xi^*_{m-1}$ gives an admissible set $y_i = u_i(x)$ on the interval $\xi_{m-1}\xi^*_m$ with $I(u) \leqq I^*(u) = 0$. Lemma 2.1 insures again the existence of ξ_m conjugate to ξ_{m-1}, with $\xi_m \leqq \xi^*_m$.

The proof of theorem 2.1 is complete, because the same type of argument combined with the separation theorem 1.4 applies also in the case of the points $\xi_{-\beta}$ preceding ξ.

Proof of theorem 2.2: The inequality (2.11) implies (2.8) and the inequalities (2.10) are already established by theorem 2.1. All we have to do is to strengthen (2.10) to (2.12).

The set $y_i = u_i(x)$ used in (2.13) does not vanish identically on $\xi\xi^*_1$. The relations (2.13) and (2.11) therefore give $I(u) < I^*(u) = 0$. This result and lemma 2.2 make the equality $\xi_1 = \xi^*_1$ impossible. We therefore actually have $\xi_1 < \xi^*_1$. Suppose now $\xi_{m-1} < \xi^*_{m-1}$ to be true $(m \leqq k)$. A similar argument to that in the proof of theorem 2.1 provides an admissible set $y_i = u_i(x)$ on $\xi_{m-1}\xi^*_m$ with $I(u) < 0$. The equality $\xi_m = \xi^*_m$ would again contradict lemma 2.2. A similar argument proves the remaining inequalities (2.12).

3. The Generalized Oscillation Theorem.

With $\Omega(x, y, y')$ defined by (1.1) and (1.2) we set up the system

$$(3.1) \qquad (d/dx)\Omega_{y_i'} - \Omega_{y_i} + A_{ik}(x, \lambda)y_k = 0, \quad (i = 1, 2, \cdots, n),$$

making on the coefficients $A_{ik}(x, \lambda)$ the following assumptions:

1. $A_{ik}(x, \lambda)$ $(i, k = 1, 2, \cdots, n)$ are supposed to be continuous functions of (x, λ) for $x_1 \leqq x \leqq x_2$, $\lambda \geqq \lambda_0$.

2. With the same λ_0 we suppose $A_{ik}(x, \lambda')u_iu_k < A_{ik}(x, \lambda'')u_iu_k$, for $x_1 \leqq x \leqq x_2$, $u_iu_i > 0$, and $\lambda_0 \leqq \lambda' < \lambda''$.

3. For every $M > 0$ arbitrarily large, there exists a number $\lambda(M)$, such that $A_{ik}(x, \lambda)u_iu_k > M$, for $x_1 \leqq x \leqq x_2$, $u_iu_i = 1$, and $\lambda > \lambda(M)$.

4. The interval x_1x_2 contains no point conjugate to x_1 with respect to the system (3.1), taken for $\lambda = \lambda_0$.

We shall prove the following

OSCILLATION THEOREM 3.1. *The assumptions* 1, 2, 3, 4, *imply the existence of a unique sequence of values* $(\lambda_0 < \lambda_1 \leq \lambda_2 < \cdots < \lambda_m < \cdots)$ *with*

$$\lim_{m\to\infty} \lambda_m = \infty, \tag{3.2}$$

and with the following property: For $\lambda = \lambda_m$ $(m = 1, 2, \cdots)$, *there is a conjugate system of points with respect to the system* (3.1), *containing exactly* $m + 1$ *points and whose first and last point are* x_1 *and* x_2 *respectively.*

Of interest is the special case

$$A_{ik}(x, \lambda) = \lambda A_{ik}(x), \tag{3.3}$$

with $A_{ik}(x)$ *continuous and* $A_{ik}(x)u_iu_k$ *positive definite on* x_1x_2. *If* x_1x_2 *contains no point conjugate to* x_1 *with respect to the system* (1.3), *then our assumptions* 1, 2, 3, *and* 4, *are fulfilled with* $\lambda_0 = 0$. *The sequence* λ_m *has in this case the further property:*

$$\sum_{m=1}^{\infty} \lambda_m^{-\alpha} \text{ converges for } \alpha > \tfrac{1}{2}, \text{ and diverges for } \alpha \leqq \tfrac{1}{2}. \tag{3.4}$$

We define the sequence of functions

$$\xi_1(\lambda), \xi_2(\lambda), \cdots, \xi_m(\lambda), \cdots \text{ for } \lambda \geqq \lambda_0, \tag{3.5}$$

in the following way: For every $m = 1, 2, 3, \cdots$, $\xi_m(\lambda)$ shall be the m-th conjugate point to x_1 with respect to the system (3.1), or $\xi_m(\lambda) = x_2$ if there is no such point.

In order to prove the continuity and monotoneity of these functions we need some lemmas.

LEMMA 3.1. *Let* $x = \phi(\xi)$ *be the conjugate point to* ξ $(x_1 \leqq \xi \leqq x_2)$ *following* ξ, *conjugate with respect to* (1.3); *and let* $\phi(\xi) = x_2$ *if there is no such point. This function* $x = \phi(\xi)$ *is a continuous and non-decreasing function of* ξ *for* $x_1 \leqq \xi \leqq x_2$, *and also properly increasing on the interval defined by* $\phi(\xi) < x_2$, *which interval may have no points.*

The monotoneity property follows at once from the separation theorem 1.4. From the same theorem and theorem 1.3, one shows at once the impossibility of $\phi(\xi) < \phi(\xi + 0)$ or $\phi(\xi - 0) < \phi(\xi)$.

LEMMA 3.2. *Let* $2\Omega(x, y, y')$ *be defined by* (1.1), (1.2). *There are always two positive constants* b *and* c, *with*

$$P_{ik}y_iy_k + 2Q_{ik}y_iy_k' + R_{ik}y_i'y_k' \geqq -b^2y_iy_i + c^2y_i'y_i', \tag{3.6}$$

for $x_1 \leqq x \leqq x_2$, *and arbitrary values* y_i *and* y_i', *and consequently also*

$$P_{ik}y_iy_k + 2Q_{ik}y_iy_k' + R_{ik}y_i'y_k' \geqq -b^2y_iy_i. \tag{3.7}$$

The quadratic form $R_{ik}y_i'y_k'$ being positive definite on x_1x_2, we know that all the principal minors of $|\, R_{ik}\,|$ will be positive on x_1x_2. For a particular $c > 0$ sufficiently small, the determinant $|\, R_{ik} - c^2\delta_{ik}\,|$ will have the same property. The constant c being fixed in this way, we consider the determinant of order $2n$

$$\begin{vmatrix} Q_{jk} & P_{jk} + b^2\delta_{jk} \\ R_{ik} - c^2\delta_{ik} & Q_{ik} \end{vmatrix}. \tag{3.8}$$

Its principal minors of order $1, 2, \cdots, n$, lying in the upper left hand corner of the matrix are positive on x_1x_2. Its principal minors of order $n+1, n+2, \cdots, 2n$, similarly situated, are polynomials in b whose terms of highest degree are respectively $|\, R_{ik} - c^2\delta_{ik}\,|\, b^2, |\, R_{ik} - c^2\delta_{ik}\,|\, b^4, \cdots, |\, R_{ik} - c^2\delta_{ik}\,|\, b^{2n}$. We certainly can choose b sufficiently large so that all these minors shall be positive for $x_1 \leqq x \leqq x_2$. This property of the determinant (3.8) insures the inequality (3.6), the quadratic form

$$2\Omega(x, y, y') + b^2y_iy_i - c^2y_i'y_i'$$

being positive definite for $x_1 \leqq x \leqq x_2$.

LEMMA 3.3. *With*

$$I(y) = \int_{x_1}^{x_2} 2\Omega(x, y, y')\,dx, \tag{3.9}$$

we define Λ *as the greatest lower bound of* $I(y)$ *in the class of sets* $y_i = y_i(x)$ *of class* D' *on* x_1x_2, *with* $y_i(x_1) = y_i(x_2) = 0$ *and*

$$\int_{x_1}^{x_2} y_iy_i\,dx = 1. \tag{3.10}$$

Let ξ_1 *be conjugate to* x_1 *with respect to* (1.3). *We shall prove that* Λ *is always finite* † *and*

1. $\Lambda < 0$, *if* $\xi_1 < x_2$,
2. $\Lambda = 0$, *if* $\xi_1 = x_2$,
3. $\Lambda > 0$, *if there is no* $\xi_1 \leqq x_2$.

† Compare A. O. Hickson, "An Application of the Calculus of Variations to Boundary Value Problems," *Transactions of the American Mathematical Society*, Vol. 31 (1929), pp. 563-579, theorem 2 on p. 570.

From (3.7) and (3.10) we get

$$I(y) = \int_{x_1}^{x_2} 2\Omega dx \geqq -b^2 \int_{x_1}^{x_2} y_i y_i dx = -b^2,$$

and therefore Λ is finite and well defined. Our statement 1 is a consequence of the necessity of Jacobi's condition for a minimum of $I(y)$ without (3.10). The second statement is equivalent to lemma 2.2.

If there is no $\xi_1 \leqq x_2$, we know already that $\Lambda \geqq 0$ (lemma 2.1) and the only difficulty is to prove that actually $\Lambda > 0$. Our assumptions are the same as in lemma 2.1. Let (u_{ik}) be the conjugate system of solutions of (2.1) introduced in the proof of lemma 2.1. Its determinant never vanishes on x_1x_2: $U(x_0, x) \neq 0$ for $x_1 \leqq x \leqq x_2$. Let us replace in (2.1) the quadratic form $2\Omega(x, y, y')$ by $2\Omega(x, y, y') - \delta y_i y_i$, with $\delta > 0$ very small. The point x_0 being fixed, $U(x_0, x)$ will become a continuous function of δ, say $U(x_0, x, \delta)$, which will also not vanish on x_1x_2 for some small $\delta > 0$. For such a δ the interval x_1x_2 certainly contains no conjugate points with respect to the altered system (2.1) (lemma 1.1), and (2.3) gives

$$\int_{x_1}^{x_2} 2\Omega(x, y, y')\, dx - \delta \int_{x_1}^{x_2} y_i y_i dx \geqq 0,$$

for every admissible set $y_i = y_i(x)$, and hence $I(y) > \delta$ for all admissible sets with (3.10). Hence $\Lambda \geqq \delta > 0$.

LEMMA 3.4. *The first function $\xi_1(\lambda)$ of the sequence* (3.5), *is a continuous and non-increasing function for $\lambda \geqq \lambda_0$, which is actually decreasing on the λ-interval on which $\xi_1(\lambda) < x_2$.*

The monotoneity properties follow from the assumption 2 of this section and the restricted comparison theorem 2.2.

To prove the continuity of $\xi_1(\lambda)$, we consider the integral

$$I(y, \lambda) = \int_{x_1}^{x_2} [2\Omega(x, y, y') - A_{ik}(x, \lambda) y_i y_k] dx, \tag{3.11}$$

the set $y_i = y_i(x)$ being an admissible set satisfying (3.10). Let $\Lambda(\lambda)$ be the greatest lower bound of $I(y, \lambda)$ in this class of sets $y_i = y_i(x)$. We have seen that $\Lambda(\lambda)$ is well defined for $\lambda \geqq \lambda_0$ (lemma 3.3). *We shall first prove that $\Lambda(\lambda)$ is continuous for $\lambda \geqq \lambda_0$*: For λ' fixed and λ variable (both $\geqq \lambda_0$) we have from (3.11)

$$I(y, \lambda) - I(y, \lambda') = \int_{x_1}^{x_2} [A_{ik}(x, \lambda') - A_{ik}(x, \lambda)] y_i y_k dx.$$

Assumption 1 of this section insures the continuity of $A_{ik}(x, \lambda)$ as a function of λ, uniformly with respect to x on $x_1 \leqq x \leqq x_2$. Therefore

$$| A_{ik}(x, \lambda') - A_{ik}(x, \lambda) | < \epsilon \text{ for } | \lambda - \lambda' | < \delta, \; x_1 \leqq x \leqq x_2, \quad (i, k = 1, 2, \cdots, n),$$

and hence

$$| I(y, \lambda) - I(y, \lambda') | \leqq \epsilon \int_{x_1}^{x_2} \sum_{i,k=1}^{n} | y_i | \cdot | y_k | \, dx$$

$$= \epsilon \int_{x_1}^{x_2} (| y_1 |^2 + \cdots + | y_n |^2) dx = \epsilon \text{ for } | \lambda - \lambda' | < \delta.$$

This implies also $| \Lambda(\lambda) - \Lambda(\lambda') | \leqq \epsilon$ for $| \lambda - \lambda' | < \delta$, and $\Lambda(\lambda)$ is therefore continuous for $\lambda \geqq \lambda_0$.

A discontinuity of the non-increasing function $\xi_1(\lambda)$ at $\lambda = \lambda'$ $(\geqq \lambda_0)$, would imply one at least of the inequalities

$$\xi_1(\lambda' + 0) < \xi_1(\lambda'), \tag{3.12}$$

$$\xi_1(\lambda') < \xi_1(\lambda' - 0). \tag{3.13}$$

We shall get a contradiction from each of these inequalities. To fix the ideas, we suppose (3.12) to hold. We choose a fixed x_3 with

$$\xi_1(\lambda' + 0) < x_3 < \xi_1(\lambda').$$

Let $\Lambda(\lambda)$ be the greatest lower bound of $I(y, \lambda)$ defined by (3.11) for the interval $x_1 x_3$ instead of $x_1 x_2$. For $\lambda > \lambda'$ we have

$$\xi_1(\lambda) < \xi_1(\lambda' + 0) < x_3$$

and hence by lemma 3.3 (case 1), applied to the integral (3.11) instead of (3.9), we get

$$\Lambda(\lambda) < 0, \text{ for } \lambda > \lambda'. \tag{3.14}$$

But $x_3 < \xi_1(\lambda')$, and the same lemma (case 3) gives

$$\Lambda(\lambda') > 0. \tag{3.15}$$

The inequalities (3.14) and (3.15) contradict the continuity of $\Lambda(\lambda)$, because the negative number $\Lambda(\lambda)$ can have, for $\lambda \to \lambda'$, no positive limit $\Lambda(\lambda')$. Quite a similar argument contradicts (3.13). Hence our lemma is proved.

In order to prove the continuity and monotoneity of the remaining

functions of the sequence (3.5) by induction, it will be convenient to have the following lemma:

LEMMA 3.5. *We define a function* $\phi(\xi, \lambda)$ *for* $x_1 \leqq \xi \leqq x_2$, $\lambda \geqq \lambda_0$, *as follows:* $x = \phi(\xi, \lambda)$ *is the conjugate point to* ξ, *following* ξ, *conjugate with respect to the system* (3.1), *or else* $\phi(\xi, \lambda) = x_2$ *if there is no such point. Let* $\xi = \xi(\lambda)$ *be a continuous and non-increasing function of* λ, *for* $\lambda \geqq \lambda_0$, *with* $x_1 \leqq \xi(\lambda) \leqq x_2$, *for* $\lambda \geqq \lambda_0$. *Then* $\psi(\lambda) = \phi(\xi(\lambda), \lambda)$ *is a continuous and non-increasing function of* λ, *for* $\lambda \geqq \lambda_0$, *and actually decreasing on the* λ*-interval defined by* $\phi(\xi(\lambda), \lambda) < x_2$, *which interval may have no points.*

The function $\phi(\xi, \lambda)$ is a function of two variables which is analogous to the function $\phi(\xi)$ of lemma 3.1, if one replaces the system (3.1) by (1.3). Lemma 3.1 insures therefore the monotoneity and continuity of $\phi(\xi, \lambda)$ as a function of ξ only when λ is fixed. Furthermore

$$\phi(x_1, \lambda) = \xi_1(\lambda),$$

and because the proof of lemma 3.4 works also for any ξ on $x_1 x_2$, instead of $\xi = x_1$, this lemma 3.4 insures the monotoneity and continuity of $\phi(\xi, \lambda)$ as function of λ only when ξ is fixed. This, of course, does not imply the continuity of $\phi(\xi, \lambda)$ as function of both variables. But the more restricted statement of our lemma follows easily as indicated in the following paragraphs.

We prove first

a) *The values* (ξ', λ') *being fixed, then for every* $\epsilon > 0$ *there is a* $\delta > 0$, *such that*

(3.16) $\quad 0 \leqq \xi' - \xi < \delta,\ 0 \leqq \lambda - \lambda' < \delta$ *imply* $0 \leqq \phi(\xi', \lambda') - \phi(\xi, \lambda) < \epsilon.$

This statement is obvious for $\xi' = x_1$. In the proof we suppose $\xi' > x_1$. We have

$$(3.17) \quad \phi(\xi', \lambda') - \phi(\xi_0, \lambda^0) = \phi(\xi', \lambda') - \phi(\xi', \lambda^0) + \phi(\xi', \lambda^0) - \phi(\xi_0, \lambda^0).$$

The function $\phi(\xi', \lambda)$ being continuous in λ at $\lambda = \lambda'$, there is a $\lambda^0 > \lambda'$ with $0 \leqq \phi(\xi', \lambda') - \phi(\xi', \lambda^0) < \epsilon/2$. Furthermore $\phi(\xi, \lambda^0)$ is continuous at $\xi = \xi'$ and one can choose a $\xi_0 < \xi'$ with $0 \leqq \phi(\xi', \lambda^0) - \phi(\xi_0, \lambda^0) < \epsilon/2$. From (3.17) it follows $0 \leqq \phi(\xi', \lambda') - \phi(\xi_0, \lambda^0) < \epsilon$. This last result and the monotoneity of $\phi(\xi, \lambda)$ in ξ and λ certainly imply (3.16) for every $\delta < \xi' - \xi_0$ and $< \lambda^0 - \lambda'$.

Now we prove

b) *The values* (ξ', λ') *being fixed, then for every* $\epsilon > 0$ *there is a* $\delta > 0$, *such that*

(3.18) $0 \leqq \xi - \xi' < \delta$, $0 \leqq \lambda' - \lambda < \delta$ *imply* $0 \leqq \phi(\xi, \lambda) - \phi(\xi', \lambda') < \epsilon$.

This statement is obvious if one or both of the equalities $\xi' = x_2$, $\lambda' = \lambda_0$ hold. In the proof we suppose $\xi' < x_2$, $\lambda' > \lambda_0$. The argument is the same as above. We have

$$(3.19)\quad \phi(\xi_0, \lambda^0) - \phi(\xi', \lambda') = \phi(\xi_0, \lambda^0) - \phi(\xi', \lambda^0) + \phi(\xi', \lambda^0) - \phi(\xi', \lambda').$$

There is a $\lambda^0 < \lambda'$ with $0 \leqq \phi(\xi', \lambda^0) - \phi(\xi', \lambda') < \epsilon/2$, and a $\xi_0 > \xi'$ with $0 \leqq \phi(\xi_0, \lambda^0) - \phi(\xi', \lambda^0) < \epsilon/2$, and therefore (3.18) holds for every $\delta < \xi_0 - \xi'$ and $< \lambda' - \lambda^0$.

The proof for our lemma follows at once. The function

$$\psi(\lambda) = \phi(\xi(\lambda), \lambda)$$

is certainly non-increasing for $\lambda \geqq \lambda_0$, and is actually decreasing on the λ-interval for which $\psi(\lambda) < x_2$ and which may have no point (for instance if $\xi(\lambda) \equiv x_2$). For λ fixed, our statement a) insures the inequality

$$0 \leqq \phi(\xi(\lambda), \lambda) - \phi(\xi(\lambda + \eta), \lambda + \eta) < \epsilon$$

for $0 \leqq \eta < \delta'$ and hence $\psi(\lambda) = \psi(\lambda + 0)$. Similarly statement b) insures $\psi(\lambda) = \psi(\lambda - 0)$, and our lemma 3.5 is proved.

LEMMA 3.6. *The functions* $\xi_m(\lambda)$, $(m = 1, 2, 3, \cdots)$, *of the sequence* (3.5) *are continuous and non-increasing functions of* λ *for* $\lambda \geqq \lambda_0$, *and actually decreasing on the* λ*-intervals for which* $\xi_m(\lambda) < x_2$.

For $m = 1$, this is the statement of lemma 3.4. The proof goes on by induction. Suppose the theorem holds for $1, 2, \cdots, m - 1$. We have

$$\xi_m(\lambda) = \phi(\xi_{m-1}(\lambda), \lambda),$$

$\phi(\xi, \lambda)$ being the function defined by lemma 3.5. By hypothesis $\xi_{m-1}(\lambda)$ is a continuous non-increasing function of λ for $\lambda \geqq \lambda_0$. Lemma 3.5 says that $\xi_m(\lambda)$ is also a continuous non-increasing function of λ for $\lambda \geqq \lambda_0$, which is actually decreasing for $\xi_m(\lambda) < x_2$. This proves our lemma by induction.

The proof of the oscillation theorem 3.1 now makes no difficulty. We have

$$(3.20)\qquad \xi_1(\lambda) \leqq \xi_2(\lambda) \leqq \cdots \leqq \xi_m(\lambda) \leqq \cdots \text{ for } \lambda \geqq \lambda_0.$$

From assumption 4 of this section, we get $\xi_1(\lambda_0) = x_2$ and (3.20) gives

$$\xi_m(\lambda_0) = x_2, \ (m = 1, 2, 3, \cdots). \tag{3.21}$$

We shall prove now

$$\lim_{\lambda\to\infty} \xi_m(\lambda) = x_1, \ (m = 1, 2, 3, \cdots). \tag{3.22}$$

To prove this we determine the constant $a > 0$ satisfying

$$P_{ik}(x)u_iu_k + 2Q_{ik}(x)u_iv_k + R_{ik}(x)v_iv_k \leqq a^2, \tag{3.23}$$

for $x_1 \leqq x \leqq x_2$, and $u_iu_i + v_iv_i = 1$. Inequality (3.23) and assumption 3 of this section give

$$2\Omega(x, y, y') - A_{ik}(x, \lambda)y_iy_k \leqq a^2(y_iy_i + y_i'y_i') - My_iy_i, \tag{3.24}$$

for $x_1 \leqq x \leqq x_2$, $\lambda > \lambda(M)$, and arbitrary values y_i, y_i'.

We shall apply the comparison theorem 2.1 for

$$2\Omega^* = (a^2 - M)y_iy_i + a^2y_i'y_i'$$

and the left hand side of (3.24) instead of 2Ω. The system (2.7) becomes

$$(d^2/dx^2)y_i + \alpha^2y_i = 0, \tag{3.25}$$

with $\alpha = a^{-1}(M - a^2)^{\frac{1}{2}}$ (supposing $M > a^2$). In order to determine the conjugate system of points defined by $x = x_1$ with respect to the system (3.25), we remark that

$$u_{ik}(x) \equiv 0 \ \ (i \neq k), \quad u_{kk}(x) = \sin \alpha(x - x_1)$$

is a conjugate system of solutions of (3.25) and hence

$$U(x_1, x) = | \, u_{ik}(x) \, | \, = (\sin \alpha(x - x_1))^n.$$

The first zero of this determinant which follows x_1 is $\xi^*_1 = x_1 + \pi/\alpha$. The m-th conjugate point of x_1, following x_1, will therefore be

$$\xi^*_m = x_1 + m\pi/\alpha.$$

The comparison theorem 2.1 gives

$$x_1 < \xi_m(\lambda) \leqq x_1 + m\pi a(M - a^2)^{-\frac{1}{2}}$$

and this implies (3.22), when we allow M to increase.

We define $\lambda_m > \lambda_0 (m = 1, 2, 3, \cdots)$ *as a value for which*

$$\xi_m(\lambda_m) = x_2, \quad \xi_m(\lambda_m + \eta) < x_2 \tag{3.26}$$

for every $\eta > 0$. Its existence follows from (3.21) and (3.22). We shall prove first that x_2 *is the m-th conjugate point of* x_1 *with respect to the system* (3.1) for $\lambda = \lambda_m$. We know that $\xi_m(\lambda_m + \eta) (< x_2)$ is the m-th conjugate point of x_1 with respect to (3.1) for $\lambda = \lambda_m + \eta$, for every $\eta > 0$. For $\lim \eta = 0$ we get the result that $\xi_m(\lambda_m) = x_2$ is the m-th conjugate point of x_1 with respect to (3.1) for $\lambda = \lambda_m$. It follows also that $x_2 = \xi_{m+1}(\lambda_{m+1})$ is the $(m+1)$-st conjugate of x_1 with respect to (3.1) for $\lambda = \lambda_{m+1}$, and hence $\xi_m(\lambda_{m+1}) < x_2 = \xi_m(\lambda_m)$. Therefore $\lambda_m < \lambda_{m+1}$ and

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m < \cdots . \tag{3.27}$$

We shall prove finally that

$$\lim_{m \to \infty} \lambda_m = \infty. \tag{3.2}$$

Suppose (3.2) does not hold and suppose that λ_m approaches Λ, as m approaches ∞. We take m sufficiently large so that $\xi_m(\Lambda) = x_2$. One therefore has $\xi_m(\Lambda) = x_2 = \xi_m(\lambda_m)$ and (3.26) implies $\Lambda \leqq \lambda_m$, which result contradicts (3.27) and the definition of Λ. Our oscillation theorem 3.1 is therefore completely proved.

We consider now the special case (3.3) and shall prove the statement (3.4). Let k and K be two positive constants satisfying

$$k^2 \leqq A_{ik}(x) u_i u_k \leqq K^2, \quad \text{for} \quad x_1 \leqq x \leqq x_2, \quad u_i u_i = 1. \tag{3.28}$$

The inequalities (3.6), (3.23), and (3.28) give

$$\begin{aligned} c^2 y_i' y_i' - (K^2\lambda + b^2) y_i y_i \leqq 2\Omega(x, y, y') \\ - \lambda A_{ik}(x) y_i y_k \leqq a^2 y_i' y_i' - (k^2\lambda - a^2) y_i y_i, \end{aligned} \tag{3.29}$$

for $x_1 \leqq x \leqq x_2$, $\lambda > 0$, and y_i, y_i' arbitrary. We shall apply the comparison theorem 2.1 twice, comparing the three differential systems which correspond to the three quadratic forms of (3.29). The system corresponding to the second form is our system (3.1), while the systems corresponding to the first and the third form are of the type (3.25). We have seen how to determine the conjugate systems of points for systems of the type (3.25). The formulas there obtained and theorem 2.1 give

$$x_1 + m\pi c(K^2\lambda_m + b^2)^{-\frac{1}{2}} \leqq \xi_m(\lambda_m) = x_2 \leqq x_1 + m\pi a(k^2\lambda_m - a^2)^{-\frac{1}{2}}$$

and hence

$$K^{-2}[m^2\pi^2c^2(x_2 - x_1)^{-2} - b^2] \leqq \lambda_m \leqq k^{-2}[m^2\pi^2a^2(x_2 - x_1)^{-2} + a^2].$$

These last inequalities show that $\sum_{m=1}^{\infty} \lambda_m^{-a}$ converges or diverges simultaneously with $\sum_{m=1}^{\infty} m^{-2a}$. Hence (3. 4) is proved.

We want to emphasize finally the difference between the theorem 3. 1 and Hickson's oscillation theorem (*loc. cit.*). Hickson established for the special case of Theorem 3. 1 the existence of an increasing sequence of characteristic numbers λ_n for each of which a certain boundary value problem has a finite number of linearly independent solutions η_i with elements vanishing at x_1 and x_2. In the preceding pages a similar sequence of characteristic numbers λ_n is found for each of which there exists a conjugate system in the sense defined above of precisely $n + 1$ points with x_1 and x_2 as its initial and final point.

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ON IDENTITIES ARISING FROM SOLUTIONS OF q-DIFFERENCE EQUATIONS AND SOME INTERPRETATIONS IN NUMBER THEORY.

By G. W. STARCHER.

Introductory. The interpretation of certain kinds of identities in number theory has, since the time of Euler, stimulated numerous independent investigations for the purpose of obtaining identities involving infinite products and series of a simple character. Often identities were discovered empirically and later verified or proved, as was the case in some of the investigations of Euler and Ramanujan. Many of the proofs of known identities depend upon a purely algebraic rearrangement of products and series. Simple and often highly ingenious proofs of certain algebraic identities have been given by means of their interpretations in number theory.* Some of the most beautiful identities known, many of them having important applications to number theory, have been demonstrated by Jacobi † by means of the theory of elliptic functions. L. J. Rogers,‡ by quite general methods, has established identities of considerable interest and beauty in themselves.

This investigation is based on a study of two linear q-difference equations with linear coefficients taken in the forms (1. 1) and (4. 1) respectively. Each of these equations has a unique solution analytic in the neighborhood of the point zero and having there the value 1, and hereafter such a solution will be said to have the property A. Obtaining various expressions for such a solution, and equating them, we have identities.

While most of the identities found appear to be new they include many interesting and classic identities as special cases of the more general identities and thus afford new proofs for them. For this reason a number of such identities are stated and often new members are added. The method of exhibiting these identities serves to relate identities that have previously appeared to be quite unrelated.

1. *Solutions of a Non-homogeneous Equation.* Consider the equation

* For example, see MacMahon, *Combinatory Analysis*, Vol. 2.

† "Theoria Evolutionis Functionum Ellipticarum," *Gesammelte Werke*, Vol. 1, pp. 141-239, and Ueber unendliche Reihen deren Exponenten zugleich in zwei verscheidenen Quadratischen Formen enthalten sind, Vol. 2, pp. 218-288.

‡ "On the Expansion of Certain Infinite Products," *Proceedings of the London Mathematical Society*, Vol. 24 (1893), pp. 337-352; *ibid.*, Vol. 25 (1894), pp. 318-343.

$$(1.1) \qquad (1+\beta x)f(x)+(\gamma+\delta x)f(qx)+\lambda+\mu x=0, \qquad |q|<1.$$

If there is a solution of (1.1) having the property A it may be written in the form

$$(1.2) \qquad f_1(x)=1+\Sigma c_\nu x^\nu,$$

where here, as in the following pages, the index ν connotes the range 1, 2, 3, $\cdots$. In order that this series satisfy (1.1) and is other than the trivial solution $f_1(x)\equiv 1$ we must have

$$(1.3) \qquad 1+\gamma+\lambda=0, \qquad \epsilon=\beta+\delta+\mu\neq 0.$$

Then we have the relations

$$-c_1(1+\gamma q)=\epsilon; \qquad -c_\nu(1+\gamma q^\nu)=c_{\nu-1}(\beta+\delta q^{\nu-1}), \qquad (\nu>1),$$

which determine the c's uniquely. Hence we have the unique solution *

$$(1.4) \qquad f_1(x)=1+\Sigma\epsilon(-x)^\nu(\beta,\delta q)_{\nu-1}(\gamma q)'_\nu,$$

which is convergent if $|\beta x|<1$.

Let us consider the possibility of expressing such a solution in each of the forms

$$(1.5) \qquad f_2(x)=1+\Sigma c_\nu x^\nu(\beta x)'_\nu,$$

$$(1.6) \qquad f_3(x)=1+\Sigma c_\nu x(\beta x)'_\nu,$$

$$(1.7) \qquad f_4(x)=1+\Sigma c_\nu x(\gamma,\delta x)_{\nu-1}, \qquad (\beta=0),$$

$$(1.8) \qquad f_5(x)=1+\Sigma c_\nu x^{[(\nu+1)/2]}(\gamma,\delta x)_{[\nu/2]-1}, \qquad (\beta=0),$$

where $[k]$ denotes the least integer $\geq k$. Clearly each of these series, if it converges, defines a function having the property A.

Substituting (1.5) into (1.1), solving for the successive c's by equating coefficients of expressions of the form $x^\nu(\beta x)'_\nu$ to zero, we have the solution

$$(1.9) \qquad f_2(x)=1+\Sigma q^{\nu(\nu-1)/2}\epsilon(\delta,-\beta\gamma q)_{\nu-1}(\gamma q)'_\nu(\beta x)'_\nu(-x)^\nu.$$

For (1.6) we consider only the case $\delta=0$, to insure convergence of series obtained on substituting into (1.1), and obtain the solution

$$(1.10) \qquad f_3(x)=1+\Sigma(-)^\nu\epsilon(\gamma q)^{\nu-1}x(\beta x)'_\nu, \qquad (\delta=0).$$

Similarly when $\beta=0$ we get the solutions

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* For convenience we shall write $(a,x)_\nu$ for $(a+x)(a+qx)\cdots(a+q^{\nu-1}x)$, $(a,x)'_\nu$ for $1/(a,x)_\nu$; $(x)_\nu$ for $(1,x)_\nu$, and $(x)'_\nu$ for $1/(x)_\nu$, etc., and throughout this paper any symbol marked prime denotes the reciprocal of the expression denoted by the unprime symbol. In particular we make $(a,x)_0=1$ and $(x)=(x)_\infty=\Pi(1+q^{\nu-1}x)$.

Any values of the parameters or of x for which a series or product is undefined or is infinite will be known as exceptional values for that series or product. Such values we exclude by enclosing them in circles of arbitrarily small, but fixed, radius.

$$(1.11)\qquad f_4(x)=1+\Sigma(-)^{\nu}\epsilon q^{\nu-1}x(\gamma,\delta x)_{\nu-1},$$

$$(1.12)\qquad f_5(x)=1+\Sigma(-x)^{\nu}\epsilon\delta^{\nu-1}\times\{q^{3\nu(\nu-1)/2}(\gamma,\delta x)_{\nu-1}(\gamma q)'_{\nu-1}-q^{\nu(3\nu-1)/2}(\gamma,\delta x)_{\nu}(\gamma q)'_{\nu}\}.$$

Clearly each of the last four series converge for all x.

2. *Case* $\lambda=\mu=0$. *The Homogeneous Case.* By (1.3), (1.1) takes the simple form

$$(2.1)\qquad (1+\beta x)f(x)=(1-\delta x)f(qx).$$

We easily obtain as a solution having the property A

$$(2.2)\qquad f_6(x)=\Pi\,[(1-\delta q^{\nu-1}x)/(1+\beta q^{\nu-1}x)]=(-\delta x)\,(\beta x)'.$$

For the rest of this section let $\beta=0$.

If in (2.1) we substitute

$$(2.3)\qquad f(x)=1+\Sigma c_{\nu}(-qx)_{\nu},$$

we obtain (writing $c_0=1$)

$$1+\Sigma c_{\nu}(1-qx)\,(-q^2x)_{\nu-1}=1+\Sigma c_{\nu}(1-q^{\nu+1}x)\,(-q^2x)_{\nu-1}-\Sigma c_{\nu-1}\delta x(-q^2x)_{\nu-1}.$$

On equating coefficients of $(-q^2x)_{\nu-1}$ we get

$$(2.4)\qquad c_{\nu}=\delta^{\nu}q^{-\nu}(-q)'_{\nu}.$$

If $|\delta|<|q|$, $f(x)$ is convergent for all x. By making $\beta=\lambda=\mu=0$ and $x=-1$ in (1.4) and comparing with (2.2) we have from (2.3) when the c's have their values given by (2.4)

$$f(0)=1+\Sigma\delta^{\nu}q^{-\nu}(-q)'_{\nu}=1/\Pi(1-q^{\nu-2}\delta)=(-q^{-1}\delta)'.$$

Hence, if $\beta=0$, $f_6(x)$ becomes

$$(2.5)\qquad f_7(x)=(-q^{-1}\delta)\{1+\Sigma\delta^{\nu}q^{-\nu}(-q)'_{\nu}(-qx)_{\nu}\},$$

since $f_7(x)$ is also a solution with property A.

Replacing q by q^2 in (2.1), $\beta=0$, we get

$$(2.6)\qquad f(x,q^2)=(1-\delta x)f(q^2x,q^2).$$

If in (2.6) we substitute

$$(2.7)\qquad f(x,q^2)=1+\Sigma c_{\nu}(-\delta q^{-1}x)_{\nu},$$

and equate the quantities which multiply the expressions $(-\delta x)_{\nu}$, $\nu=1,2,\cdots$, we have

$$c_1=q/(1-q),\qquad c_{\nu}=c_{\nu-1}q^{\nu}/(1-q^{\nu}),\qquad (\nu>1),$$

and hence the solution *

$$f(x, q^2) = 1 + \Sigma q^{\nu(\nu+1)/2}(-\delta q^{-1}x)_\nu(-q)'_\nu.$$

But $f(0, q^2) = (q)$ by comparison with (2.2) and (1.9). Then we have the solution with property A,

$$(2.8) \qquad f_7(x) = (q)'\{1 + \Sigma q^{\nu(\nu+1)/2}(-q)'_\nu(-\delta q^{-1}x)_\nu\},$$

which by (2.2) is the same as $\Pi(1 - \delta q^{2\nu-2}x)$.

3. *Applications and Interpretations.* We replace $f(x)$ by the symbol $f(\beta x, \gamma, \delta x, \mu x, q)$, so that f_3 becomes $f(\beta x, \gamma, 0, \mu x, q)$, f_4 and f_5 become $f(0, \gamma, \delta x, \mu x, q)$, f_6 becomes $f(\beta x, -1, \delta x, 0, q)$, f_7 becomes $f(0, -1, \delta x, 0, q)$, and $f_7(x, q^2)$ becomes $f(0, -1, \delta x, 0, q^2)$. From identities that arise from equating various expressions for the same function as obtained in the preceding sections we have †

$$(3.1) \quad f(\beta x, -1, \delta x, 0, q) = (-\delta x)(\beta x)' = 1 + \Sigma(\beta, \delta)_\nu(-q)'_\nu(-x)^\nu$$
$$= 1 + \Sigma q^{\nu(\nu-1)/2}(\delta, \beta)_\nu(-q)'_\nu(\beta x)'_\nu(-x)^\nu.$$

From (3.1) we obtain

$$(3.2) \quad f(qx, -1, qx, 0, q) = (-qx)(qx)' = 1 + 2\Sigma q^\nu(q)_{\nu-1}(-q)'_\nu(-x)^\nu$$
$$= 1 + 2\Sigma q^{\nu(\nu+1)/2}(q)_{\nu-1}(-q)'_\nu(qx)'_\nu(-x)^\nu,$$

$$(3.3) \quad f(\pm qx, -1, \pm qx^2, 0, q) = (\mp qx^2)(\pm qx)' = 1 + \Sigma(\mp qx)^\nu(x)_\nu(-q)'_\nu$$
$$= 1 + \Sigma q^{\nu(\nu+1)/2}(x, 1)_\nu(-q)'_\nu(\pm qx)'_\nu(\mp x)^\nu,$$

$$(3.4) \quad f(q, -1, qx, 0, q) = (-qx)(q)' = 1 + \Sigma(-q)^\nu(x)_\nu(-q)'_\nu$$
$$= 1 + \Sigma(-)^\nu q^{\nu(\nu+1)/2}(x, 1)_\nu(-q)'_\nu(q)'_\nu,$$

$$(3.5) \quad f(qx, -1, -q, 0, q) = (q)(qx)' = 1 + \Sigma(-q)^\nu(x, -1)_\nu(-q)'_\nu$$
$$= 1 + \Sigma(-)^\nu q^{\nu(\nu+1)/2}(x, -1)_\nu(-q)'_\nu(qx)'_\nu.$$

These examples illustrate the variety of identities one may obtain in this way. Note the formulae similar to the last three given by the three corresponding expressions for $f(\pm qx^2, -1, \mp qx, 0, q)$, $f(-q, -1, qx, 0, q)$, and $f(qx, -1, q, 0, q)$ respectively. Note also the formulae obtained from those above by replacing x by $-x$, by ± 1, or by certain functions of q. For example

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* Since this was written my attention has been called to C. H. Ashton's dissertation, *Die Heineschen O-Functionen und ihre Anwendungen auf die Elliptische Functionen* (Munich, 1909). He has studied the same function in another form; also my $f(0, -1, qx, 0, q)$ is identical with his $0_{00}(z)$ with $x = e^{2\pi i z/w_1}$.

† Essentially the same formulae have been given by Cauchy, *Oeuvres* (1) VIII, pp. 42-50.

$$(3.6)\quad f(q,-1,q,0,q)=(-q)(q)'=1+2\Sigma(-q)^{\nu}(q)_{\nu-1}(-q)'_{\nu}$$
$$=1+2\Sigma(-)^{\nu}q^{\nu(\nu+1)/2}(q)_{\nu-1}(-q)'_{\nu}(q)'_{\nu},$$

and by the theory of elliptic functions the second member is ϑ_4. Similarly we find

$$\vartheta_2=2q^{1/4}\pi(1+q^{2\nu})^2(1-q^{2\nu})$$
$$=2q^{1/4}\Pi[(1-q^{4\nu})/(1-q^{4\nu-2})]=2q^{1/4}f(-q^2,-1,q^4,0,q^4).$$

Thus we have two new expressions for each of the Theta constants ϑ_2, ϑ_3, and ϑ_4. Another example of such identities in q is given by

$$(3.7)\quad f(-q,-1,-q,0,q)=(q)(-q)'$$
$$=1+2\Sigma q^{\nu}(q)_{\nu-1}(-q)'_{\nu}=1+2\Sigma q^{\nu(\nu+1)/2}(q)_{\nu-1}[(-q)'_{\nu}]^2.$$

Again we have *

$$(3.8)\quad f(-qx,-1,q^{j+1}x,0,q)=1/\prod_{\nu=1}^{j}(1-q^{\nu}x)=1+\Sigma(qx)^{\nu}(-q^{j})_{\nu}(-q)'_{\nu}$$
$$=1+\sum_{\nu=1}^{j}q^{\nu^2}(-q^{j-\nu+1})_{\nu}(-q)'_{\nu}(-qx)'_{\nu}x^{\nu}=1+\Sigma_{\nu,\mu}P(\nu,\mu,\not>j)q^{\nu}x^{\mu},$$

where the symbol $P(\nu,\mu,\not>j)$ represents the number of partitions of ν into exactly μ summands none greater than j; and

$$(3.9)\quad f(q^{j+1}x,-1,qx,0,q)=\prod_{\nu=1}^{j}(1+q^{\nu}x)=1+\sum_{\nu=1}^{j}q^{\nu(\nu+1)/2}(-q^{j-\nu+1})_{\nu}(-q)'_{\nu}x^{\nu}$$
$$=1+\Sigma q^{\nu(\nu+1)/2}(-q^{j})_{\nu}(-q)'_{\nu}(q^{j+1}x)'_{\nu}x^{\nu}=1+\Sigma_{\nu,\mu}P^{i}(\nu,\mu,\not>j)q^{\nu}x^{\mu},$$

where the superscript i on the symbol P indicates that the parts are distinct. We may observe from the second members of (3.8) and (3.9) the theorem,

$$P(\nu,k,\not>j)=P^{i}(\nu+\tfrac{1}{2}k(k-1),k,\not>j+k-1).$$

From the expressions in § 2 we have

$$(3.10)\quad f(-qx,-1,0,0,q)=1/\Pi(1-q^{\nu}x)=1+\Sigma(qx)^{\nu}(-q)'_{\nu}$$
$$=1+\Sigma q^{\nu^2}(-q)'_{\nu}(-qx)'_{\nu}x^{\nu}=1+\Sigma q^{\nu}x(-qx)'_{\nu}$$
$$=1+\Sigma_{\nu,\mu}P(\nu,\mu,*)q^{\nu}x^{\mu}.$$

This is one of the fundamental identities in the theory of partitions. The symbol $P(\nu,\mu,*)$ represents the number of partitions of ν into μ positive summands. Replacing x by $q^{-1}x$ we have a similar identity involving the function which enumerates the partitions of ν into μ or fewer summands unrestricted in magnitude. By equating coefficients of $q^{\nu}x^{k}$ in these identities

* Cf. MacMahon, *Combinatory Analysis*, Vol. 2, Ch. II. MacMahon demonstrates the identity (3.8) and the second and third members of (3.9).

we obtain identities in certain partition functions. When $x=1$ by the 2nd and 3rd members of (3.10) we have an identity due to Euler, and probably the most interesting identity in the theory of partitions. Generalizations are obtained by writing q^{m-n} for x and q^n for q.

Again we have *

$$(3.11)\quad f(0,-1,qx,0,q)=\Pi(1-q^\nu x)=1+\Sigma q^{\nu(\nu+1)/2}(-q)_\nu'(-x)^\nu$$
$$=1-\Sigma q^\nu x(-qx)_{\nu-1}$$
$$=1+\Sigma q^{\nu(3\nu-1)/2}(-x)^\nu(1-q^{2\nu}x)(-qx)_{\nu-1}(-q)'_\nu,$$

which for $x=1$ gives

$$(3.12)\quad f(0,-1,q,0,q)=\Pi(1-q^\nu)=1+\Sigma(-)^\nu q^{\nu(\nu+1)/2}(-q)'_\nu$$
$$=1-\Sigma q^\nu(-q)_{\nu-1}=1+\Sigma(-)^\nu q^{\nu(3\nu-1)/2}(1+q^\nu)$$
$$=\sum_{n=-\infty}^{\infty}(-1)^n q^{(3n^2+n)/2}.$$

The second and last members above constitute a celebrated identity due to Euler, of which many proofs have been given. If in (3.11) we replace x by $-x$ and equate coefficients of x^k we have the almost obvious theorem $P^i(\nu+\frac{1}{2}k(k-1),k,*)=P(\nu,k,*)$. Similar theorems come from (3.12).

Using the fact that $\Pi(1+q^\nu)^{-1}=\Pi(1-q^{2\nu-1})$ we have †

$$(3.13)\quad (-q;q^2)f(-q,-1,qx,0,q^2)=f(0,-1,qx,0,q^2)=\Pi(1-q^{2\nu-1}x)$$
$$=(-q;q^2)\{1+\Sigma q^\nu(-x;q^2)_\nu(-q^2;q^2)'_\nu\}$$
$$=(-q;q^2)\{1+\Sigma(-)^\nu q^{\nu^2}(x^{-1};q^2)_\nu(-q)'_{2\nu}\}$$
$$=(-q;q^2)\{1+\Sigma q^{\nu(\nu+1)/2}(-x)_\nu(-q)'_\nu\},$$

which, by (3.11) with q replaced by q^2 and x by $q^{-1}x$,

$$=1+\Sigma q^{\nu^2}(-x)^\nu(-q^2;q^2)'_\nu=1-\Sigma q^{2\nu-1}x(-qx;q^2)_\nu$$
$$=1+\Sigma q^{3\nu^2-2\nu}(-x)^\nu(1-q^{4\nu-1}x)(-qx;q^2)_{\nu-1}(-q^2;q^2)'_\nu.$$

We can write

$$f(0,-1,qx,0,q^2)=\Pi[1-q^\nu x)/\Pi(1-q^{2\nu}x)]$$
$$=f(0,-1,qx,0,q)\ f(-q^2x,-1,0,0,q^2).$$

Each of these can be expressed as infinite series thus giving six additional, equally simple, expressions for the function in (3.13).

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* The identity of the 2nd and 3rd members is well known. That of the 2nd and last member has been verified by E. Netto. See *Lehrbuch der Combinatorik*, pp. 163-65.

† The symbol $(x;q^k)_\nu$ is written for $(1+x)(1+q^kx)(1+q^{2k}x)\cdots(1+q^{\nu k-k}x)$, accordingly we write $(x;q^k)$ for $\Pi(1+q^{k(\nu-1)}x)$. The third and sixth members of (3.13) constitute an identity due to Gauss and verified by Lebesgue. See *Liouville's Journal* (1), 5, p. 47.

From the first five members above we have

$$(3.14)\quad f(0,-1,q^3,0,q^2)/f(0,-1,q,0,q^2)=(-q^2;q^2)(-q;q^2)'$$
$$=f(-q,-1,q^2,0,q^2)=1+\Sigma q^{\nu}(-q;q^2)_{\nu}(-q^2)'_{\nu}$$
$$=1-\Sigma(-)^{\nu}q^{\nu^2+\nu-1}(-q^2;q^2)'_{\nu}(1-q)/(1-q^{2\nu-1})=1+\Sigma q^{\nu(\nu+1)/2}.$$

Similarly we see that

$$(3.15)\quad (-q^2;q^2)f(-q^2,-1,q^2x,0,q^2)=f(0,-1,q^2x,0,q^2)$$
$$=\Pi(1-q^{2\nu}x)=(-q^2;q^2)\{1+\Sigma q^{2\nu}(-x;q^2)_{\nu}(-q^2;q^2)'_{\nu}\}$$
$$=(-q^2;q^2)\{1+\Sigma(-)^{\nu}q^{\nu(\nu+1)}(x^{-1};q^2)_{\nu}(-q^4;q^4)'_{\nu}$$
$$=(-q;q^2)\{1+\Sigma q^{\nu(\nu+1)/2}(-qx)_{\nu}(-q)'_{\nu}\}.$$

By (3.11) we get other equally simple expressions for this function, and since

$$f(0,-1,q^2x,0,q^2)=f(0,-1,qx,0,q)\cdot f(-qx,-1,0,0,q^2)$$

we may, as above, add still other members to (3.15).

By previous results we get

$$(3.16)\quad f(x,0,\delta x,(\lambda-1)(1+\delta)x,q)=1+\Sigma(-x)^{\nu}\lambda(\delta)_{\nu+1}$$
$$=1+\Sigma q^{\nu(\nu-1)/2}\lambda\delta^{\nu-1}(1+\delta)(-x)^{\nu}(x)'_{\nu}.$$

In particular

$$(3.17)\quad f(-qx,0,-qx,0,q)=1+2\Sigma(qx)^{\nu}(q)_{\nu-1}=1+2\Sigma q^{\nu(\nu+1)/2}x^{\nu}(-qx)'_{\nu}.$$

These identities can be given interpretations in terms of the partition functions.

It is easy to verify that $f(-qx,0,-qx,-(x-2)qx,q)$ satisfies

$$(1-qx)g(x)-xg(qx)-1+(1+q)x-x^2=0,$$

which obviously has a solution with the property A expressible in the forms

$$(3.18)\quad g(x)=1+qx^2+q^2(1+q)x^3+q^3(1+q)(1+q^2)x^4+\cdots$$
$$=1+qx^2(-qx)'_2+q^6x^4(-qx)'_4+\cdots,$$

and this by previous results

$$=f(-qx,0,-qx,-(x-2)qx,q)=1+\Sigma q^{\nu(\nu+1)/2}(-qx)'_{\nu}x^{\nu+1}.$$

Similarly we may obtain other identities; e. g. those obtained by solving the equation satisfied by $h(x)=g(x)+x$.

By identities in § 2 we obtain

$$(3.19)\quad f(-qx,-\lambda,0,(1-\lambda)qx,q)=1+\Sigma\lambda q^{\nu}x^{\nu}(-\lambda q)'_{\nu}$$
$$=1+\Sigma(\lambda x)^{\nu}q^{\nu^2}(-\lambda q)'_{\nu}(-qx)'_{\nu}=1+\Sigma\lambda^{\nu}q^{\nu}x(-qx)'_{\nu}.$$

Equating coefficients of $q^a\lambda^b x^c$ as they appear in the 2nd and 4th members we have the theorem

$$P(a-c,b-1,\not> c)=P(a-b,c-1,\not> b),$$

where $P(0, 0, \nmid\!\succ \gamma) = 1$, otherwise $P(\alpha, \beta, \nmid\!\succ \gamma) = 0$ when α or β is 0. From this we have as corollaries

$$P(a, c, \nmid\!\succ c) = P(a-1, c-1, \nmid\!\succ c+1),$$
$$P(a, b, \nmid\!\succ c) = P(a+c-b-1, c-1, \nmid\!\succ b+1).$$

By (3.19) we have the interesting identities

$$(3.20)\quad f(-qx, -x, 0, (1-x)qx, q) = 1 + \Sigma q^{\nu^2} x^{2\nu} [(-qx)'_\nu]^2,$$

$$(3.21)\quad f(-qx, x, 0, (1+x)qx, q) = 1 - \Sigma q^{\nu} x^{\nu+1} (qx)'_\nu = 1 + \Sigma (-)^\nu q^{\nu^2} x^{2\nu} (-q^2x^2; q^2)'_\nu = 1 + \Sigma (-q)^\nu x^{\nu+1} (-qx)'_\nu.$$

When $|x| < 1$ we have

$$(3.22)\quad f(x, \gamma, 0, 0, q) = 1 + \Sigma (-x)^\nu (\gamma q)'_\nu = 1 + \Sigma (-)^\nu \gamma^{\nu-1} q^{\nu-1} x (x)'_\nu = 1 - \Sigma \gamma^{\nu-1} q^{\nu(\nu-1)} x^\nu (\gamma q)'_\nu (x)'_\nu.$$

This gives

$$(3.23)\quad f(-qx, -x, 0, 0, q) = 1 + \Sigma (qx)^\nu (-qx)'_\nu = 1 + \Sigma q^{\nu^2} x^{2\nu-1} [(-qx)'_\nu]^2.$$

Again from § 2 we have

$$(3.24)\quad f(\beta x, \gamma, \beta\gamma x, \mu x, q) = 1 + \Sigma \beta^{\nu-1} (\beta + \beta\gamma + \mu)(-x)^\nu / (1 + \gamma q^\nu) = 1 + \Sigma q^{\nu(\nu-1)/2} (\beta\gamma)^{\nu-1} (\beta + \beta\gamma + \mu)(-q)_{\nu-1} (\gamma q)'_\nu (\beta x)'_\nu (-x)^\nu.$$

From this we get

$$(3.25)\quad f(q^2, -q^{-1}, -q, -q^2 - 3q, q^2) = 1 + \frac{4q}{1-q} - \frac{4q^3}{1-q^3} + \frac{4q^5}{1-q^5} - \cdots$$
$$= 1 + \frac{4q}{(1-q)(1+q^2)} + \frac{4q^4(1-q^2)}{(1-q)(1-q^3)(1+q^2)(1+q^4)} + \cdots.$$

The second member, by the theory of elliptic functions, $= \vartheta_3^2$, and by a previous result $= \{f(-q, -1, -q, 0, -q)\}^2$. Similarly we have

$$(3.26)\quad f(q, -q^{-1}, -1, 1-q-4q^{1/2}, q^2) = \frac{4q^{1/2}}{1-q} f(q, -q, -q^2, 0, q^2)$$
$$= \frac{4q^{1/2}}{1-q} - \frac{4q^{3/2}}{1-q^3} + \frac{4q^{5/2}}{1-q^5} \cdots = \frac{4q^{1/2}}{1-q^2} + \frac{4q^{5/2}(1-q^2)}{(1-q^2)(1-q^6)} + \frac{4q^{13/2}(1-q^2)(1-q^4)}{(1-q^2)(1-q^6)(1-q^{10})} + \cdots$$
$$= \frac{4q^{1/2}}{1-q} - 4\Sigma q^{\nu^2+\nu-1/2} (-q)_{2\nu-1} (-q^2; q^4)'_\nu \, (-q^3; q^2)'_\nu = \vartheta_2^2 = 4q^{1/2}\{f(-q^2, -1, q^4, 0, q^4)\}^2.$$

Also we have an identity involving the series of Clausen *

$$(3.27)\qquad f(-q,-1,q,-q,q)=1+\Sigma q^{\nu}/(1-q^{\nu})$$
$$=1+\Sigma(-)^{\nu+1}q^{\nu(\nu+1)/2}(-q)'_{\nu}/(1-q^{\nu}).$$

Many other examples of identities involving $f(\beta x,\gamma,\delta x,\mu x,q)$ for particular values of the parameters might be given. Often such identities imply identities of interest having many terms involving only simple series and products; e. g. comparing (3.10), (3.11), and (3.17) we have

$$(3.28)\qquad \{f(q,-1,0,0,q)\}^{-1}=f(0,-1,-q,0,q)=f(-q,0,-q,q,q)$$
$$=f(-q,-1,0,0,q^2)=\{f(0,-1,q,0,q^2)\}^{-1},$$

which implies

$$(3.29)\qquad \{f(q,-1,0,0,q)\}^{-1}=\Pi(1+q^{\nu})=\Pi\,(1-q^{2\nu-1})\ =1+\Sigma q^{\nu}(-q^2;q^2)'_{\nu}$$
$$=1+\Sigma q^{2\nu^2-\nu}(-q)'_{2\nu}=1+\Sigma q^{2\nu-1}(-q;q^2)'_{\nu}\ =1+\Sigma q^{\nu(\nu+1)/2}(-q)'_{\nu}$$
$$=1+\Sigma q^{\nu(3\nu-1)/2}\,(1+q^{2\nu})(q)_{\nu-1}(-q)'_{\nu}=1+\Sigma q^{\nu}(q)_{\nu-1}$$
$$=\{1+\Sigma q^{\nu(\nu+1)/2}\}/\{1+\Sigma(-)^{\nu}q^{3\nu^2-\nu}(1+q^{2\nu})\}=\text{etc.}$$

4. *The Homogeneous Linear Second Order Equation.* Consider the equation

$$(4.1)\qquad \phi(x)+(\alpha+\beta x)\phi(qx)+(\gamma+\delta x)\phi(q^2x)=0,$$

where $1+\alpha+\gamma=0$ and $\beta+\delta\neq 0$. A unique power series solution with property A is found to be

$$(4.2)\qquad \phi(x)=1+\Sigma q^{\nu(\nu-1)/2}\,(\beta,\delta)_{\nu}(-q)'_{\nu}(-\gamma q)'_{\nu}(-x)_{\nu},$$

where we have written

$$1+\alpha q^{\nu}+\gamma q^{2\nu}=1-q^{\nu}-\gamma q^{\nu}+\gamma q^{2\nu}=(1-q^{\nu})\,(1-\gamma q^{\nu}).$$

The series is convergent for all x.

When $\beta\neq 0$ let

$$(4.3)\qquad \phi(x)=\Pi(1-\beta q^{\nu-1}x)\Phi(x).$$

Then $\Phi(x)$ satisfies the q-difference equation

$$(4.4)\qquad \Phi(x)+\frac{\alpha+\beta x}{1-\beta x}\,\Phi(qx)+\frac{\gamma+\delta x}{(1-\beta x)\,(1-\beta qx)}\,\Phi(q^2x)=0.$$

If there is a solution of (4.4) expressible in the form

$$(4.5)\qquad \Phi(x)=1+\Sigma c_{\nu}(-\beta x)'_{\nu}x^{\nu},$$

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* *Journal für reine und angewandte Mathematik*, Vol. 3 (1828), p. 95. He states that the second member above

$$=q\left(\frac{1+q}{1-q}\right)+q^4\left(\frac{1+q^2}{1-q^2}\right)+q^9\left(\frac{1+q^3}{1-q^3}\right)+\cdots=\Sigma\tau(\nu)q^{\nu},$$

where $\tau(\nu)$ is the number of divisors of ν.

we find, on substituting into (4.4) and equating coefficients of the expressions $x^\nu(\beta x)'_\nu$ to zero,

$$c_1 = -(\delta+\beta\gamma q)/(1-q)(1-\gamma q),$$
$$c_\nu = -c_{\nu-1}q^{2\nu-2}(\delta+\beta\gamma q^\nu)/(1-q^\nu)(1-\gamma q^\nu).$$

Hence we have as a solution of (4.4)

$$\Phi(x) = 1 + \Sigma q^{\nu(\nu-1)}(\delta, \beta\gamma q)_\nu(-q)'_\nu(-\gamma q)'_\nu(-\beta x)'_\nu(-x)^\nu. \tag{4.6}$$

When $\beta = 0$ it can be shown that the equation satisfied by $\psi(x)$, where

$$\psi(x) = \Pi(1-\delta q^{\nu-1}x)^{-1}\phi(x), \tag{4.7}$$

does not have a solution of the form

$$\psi(x) = 1 + \Sigma c_\nu(-\delta x)'_\nu x^\nu, \tag{4.8}$$

in which the c's are simply determined, except when $\gamma = 0$. In this case $\psi(x)$ satisfies the equation

$$\psi(x) - \frac{1}{1-\delta x}\psi(qx) + \frac{\delta x}{(1-\delta x)(1-\delta qx)}\psi(q^2x) = 0, \tag{4.9}$$

having the solution

$$\psi(x) = 1 + \Sigma(-)^\nu q^{3\nu^2-2\nu}(\delta x)^{2\nu}(-q^2; q^2)'_\nu(-\delta x)'_{2\nu}. \tag{4.10}$$

When $\beta = \gamma = 0$ let

$$\Psi(x) = \Pi(1-q^\nu x)\phi(-qx\delta^{-1}). \tag{4.11}$$

Then $\Psi(x)$ satisfies the equation

$$\Psi(x) - (1-qx)\Psi(qx) - qx(1-qx)(1-q^2x)\Psi(q^2x) = 0. \tag{4.12}$$

If this equation has a solution expressible as a series with constant term 1, we find after substituting into (4.12), the expression multiplying the constant 1 is

$$1-(1-qx)-qx(1-qx)(1-q^2x) = 0\cdot x + q^2x^2 + q^3x^2(1-qx).$$

This suggests the series

$$\Psi(x) = 1 + \Sigma\{c_{2\nu-1}x^{2\nu-1}(-qx)_{\nu-1} + c_{2\nu}x^{2\nu}(-qx)_{\nu-1} + c'_{2\nu}x^{2\nu}(-qx)_\nu\}.$$

Substituting into (4.12) and computing the successive c's we have

$$c_1 = 0, \qquad c_2 = q^2, \qquad c'_2 = -q^3/(1-q), \cdots$$

and in general

$$c'_{2\nu} = c'_{2\nu}q^{2\nu} + c_{2\nu}q^{2\nu} - c'_{2\nu-2}q^{5\nu-2} + c_{2\nu-1}q^{4\nu-1}, \tag{4.13}$$
$$c_{2\nu-1} = -c'_{2\nu-2}q^{3\nu-2} + c_{2\nu-1}q^{2\nu-1} + c'_{2\nu-2}q^{4\nu-3} - c_{2\nu-3}q^{5\nu-5} + c_{2\nu-2}q^{4\nu-3}, \tag{4.14}$$
$$c_{2\nu} = -c_{2\nu-1}q^{3\nu-1} - c'_{2\nu-2}q^{5\nu-3} - c_{2\nu-2}q^{5\nu-3}. \tag{4.15}$$

Clearly we may solve (4.13) and (4.15) for $c_{2\nu}$ or $c'_{2\nu}$ in terms of c's with no subscript greater than $2\nu-1$ and hence for $c_{2\nu-2}$ and $c'_{2\nu-2}$ in terms of c's with subscripts not greater than $2\nu-3$. By repeated substitution for the c's with even subscripts in (4.14) we can ultimately show

$$c_{2\nu-1} = f(c_{2\nu-3}, c_{2\nu-5}, \cdots, c_1) + k(q).$$

Since $c_1 = 0$, $k(q) = 0$ and hence $c_3 = c_5 = \cdots = 0$. Then (4.13), (4.14), and (4.15) become

$$\begin{aligned} c'_{2\nu} &= c'_{2\nu}q^{2\nu} + c_{2\nu}q^{2\nu} - c'_{2\nu-2}q^{5\nu-2}, \\ 0 &= -c'_{2\nu-2}q^{3\nu-2} + c'_{2\nu-2}q^{4\nu-3} + c_{2\nu-2}q^{4\nu-3}, \\ c_{2\nu} &= -c'_{2\nu-2}q^{5\nu-3} - c_{2\nu-2}q^{5\nu-3}, \end{aligned}$$

from which we easily obtain

$$c_{2\nu} = (-1)^\nu q^{\nu(5\nu-1)/2}/\{(1-q)(1-q^2)\cdots(1-q^{\nu-1})\},$$

and

$$c'_{2\nu} = (-1)^\nu q^{\nu(5\nu-1)/2}/\{(1-q)\cdots(1-q^\nu)\}.$$

Hence we have * (if $\beta = \gamma = 0$)

$$\begin{aligned} (4.16)\quad \Psi(x) &= \Pi(1-q^\nu x)\phi(-qx\delta^{-1}) \\ &= 1 + \Sigma(-)^\nu\{q^{\nu(5\nu-1)/2}x^{2\nu}(-qx)_{\nu-1}(-q)'_{\nu-1} + q^{\nu(5\nu-1)/2}x^{2\nu}(-qx)_\nu(-q)'_\nu\}. \end{aligned}$$

In particular we have

$$\begin{aligned} (4.17)\quad \Psi(1) &= 1 + \Sigma(-)^\nu q^{\nu(5\nu-1)/2}(1+q^\nu) \\ &= \Pi(1-q^{5\nu})(1-q^{5\nu-3})(1-q^{5\nu-2}), \end{aligned}$$

$$\begin{aligned} (4.18)\quad \Psi(q) &= 1 + \Sigma(-)^\nu q^{\nu(5\nu-3)/2}(1+q^{3\nu}) \\ &= \Pi(1-q^{5\nu})(1-q^{5\nu-4})(1-q^{5\nu-1})/(1-q). \end{aligned}$$

The last members come from a well known identity due to Jacobi.

We return to (4.4) and seek a solution in the form

$$(4.19)\qquad \Phi(x) = 1 + \Sigma c_\nu(-\beta x)'_\nu,$$

where we make $|\delta/\beta| < |q|$ to insure convergence of series obtained after substituting in (4.4). Equating coefficients of $(-\beta x)'_\nu$ for $\nu = 1, 2, 3, \cdots$, we have finally the solution

$$(4.20)\qquad \Phi(x) = 1 + \Sigma(-\beta^{-1})^\nu(\delta, \beta\gamma q)_\nu(-q)'_\nu(-\beta x)'_\nu.$$

By (3.1)

$$\Phi(0) = 1 + \Sigma(-\beta^{-1})^\nu(\delta, \beta\gamma q)_\nu(-q)'_\nu = \Pi[1-\gamma q^\nu)/(1+\delta\beta^{-1}q^{\nu-1})].$$

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* This result with the power series representation of $\phi(-qx\delta^{-1})$ for $\beta = \gamma = 0$ gives an identity verified by Ramanujan (*Collected Papers* No. 26), and of which many proofs have been given.

Hence a solution of (4.4) with property A is

$$(4.21)\quad \Phi(x)=(\delta\beta^{-1})(-\gamma q)'\{1+\Sigma(-\beta^{-1})^{\nu}(\delta,\beta\gamma q)_{\nu}(-q)'_{\nu}(-\beta x)'_{\nu}\},$$

where $|\delta/\beta|<|q|$. But by analytic continuation we may extend the range of validity of the series and require only that $|\delta/\beta|<1$, which is necessary for the convergence of the series.

If we write

$$\phi(q\delta x^2, q^2)=\Pi(1+\delta q^{\nu-1}x)\Phi_1(x),$$

and solve the equation satisfied by $\Phi_1(x)$, we obtain the solution

$$(4.22)\quad \Phi_1(x)=1+\Sigma(-\delta x)^{\nu}q^{\nu(\nu-1)/2}(-\gamma q;q^2)_{\nu}(-q)'_{\nu}(-\gamma q)'_{\nu}(\delta x)'_{\nu},$$

provided $\beta=0$.

On writing $\phi(\beta x,\gamma,\delta x,q)$ for $\phi(x)$ we may summarize the main results of this section in the following identities:

$$(4.23)\quad \begin{aligned}\phi(\beta x,\gamma,\delta x,q)&=1+\Sigma q^{\nu(\nu-1)/2}(\beta,\delta)_{\nu}(-q)'_{\nu}(-\gamma q)'_{\nu}(-x)^{\nu}\\&=(-\beta x)\{1+\Sigma q^{\nu(\nu-1)}(\delta,\beta\gamma q)_{\nu}(-q)'_{\nu}(-\gamma q)'_{\nu}(-\beta x)'_{\nu}(-x)^{\nu}\}\\&=(-\beta x)(\delta\beta^{-1})(-\gamma q)'\{1+\Sigma(-\beta)^{\nu}(\delta,\beta\gamma q)_{\nu}(-q)'_{\nu}(-\beta x)'_{\nu}\},\end{aligned}$$

where the last member is present only when $|\delta/\beta|<1$,

$$(4.24)\quad \begin{aligned}&\phi(0,\gamma,q\delta^2x^2,q^2)\\&\quad=(\delta x)\{1+\Sigma q^{\nu(\nu-1)/2}(-\delta x)^{\nu}(-\gamma q;q^2)_{\nu}(-q)'_{\nu}(-\gamma q)'_{\nu}(\delta x)'_{\nu}\},\end{aligned}$$

$$(4.25)\quad \begin{aligned}&\phi(0,0,\delta x,q)\\&\quad=(-\delta x)\{1+\Sigma q^{3\nu^2-2\nu}(-\delta^2x^2)^{\nu}(-q^2;q^2)'_{\nu}(-\delta x)'_{2\nu-1}\},\end{aligned}$$

$$(4.26)\quad \begin{aligned}&\phi(0,0,-qx,q)\\&\quad=(-qx)'\{1+\Sigma(-)^{\nu}q^{\nu(5\nu-1)/2}x^{2\nu}(1-q^{2\nu}x)(-qx)_{\nu-1}(-q)'_{\nu}\}.\end{aligned}$$

5. *Applications.* From (4.23) we have

$$(5.1)\quad \begin{aligned}\phi(qx,\gamma,0,q)&=1+\Sigma q^{\nu(\nu+1)/2}(-q)'_{\nu}(-\gamma q)'_{\nu}(-x)^{\nu}\\&=(-qx)\{1+\Sigma q^{\nu(3\nu+1)/2}(-q)'_{\nu}(-\gamma q)'_{\nu}(-qx)'_{\nu}(-\gamma x)^{\nu}\}\\&=(-qx)(-q\gamma)'\{1+\Sigma q^{\nu(\nu+1)/2}(-q)'_{\nu}(-qx)'_{\nu}(-\gamma)^{\nu}\},\end{aligned}$$

from which we easily obtain

$$(5.2)\quad \begin{aligned}&f(0,-1,-q^2x,0,q^2)\phi(-qx,-x,0,q^2)\\&\quad=(q^2x;q^2)\{1+\Sigma q^{\nu^2}(-q^2;q^2)'_{\nu}(q^2x;q^2)'_{\nu}x^{\nu}\}\\&\quad=(qx)\{1+\Sigma(-)^{\nu}q^{3\nu^2}(-q^2;q^2)'_{\nu}(qx)'_{2\nu}x^{2\nu}\}\\&\quad=(qx;q^2)\{1+\Sigma q^{\nu^2+\nu}(-q^2;q^2)'_{\nu}(qx;q^2)'_{\nu}x^{\nu}\},\end{aligned}$$

$$(5.3)\quad \begin{aligned}\phi(qx,1,0,q)&=1+\Sigma q^{\nu(\nu+1)/2}(-x)^{\nu}[(-q)'_{\nu}]^2\\&=(-qx)\{1+\Sigma q^{\nu(3\nu+1)/2}[(-q)'_{\nu}]^2(-qx)'_{\nu}(-x)^{\nu}\}\\&=(-qx)(-q)'\{1+\Sigma(-)^{\nu}q^{\nu(\nu+1)/2}(-q)'_{\nu}(-qx)'_{\nu}\}.\end{aligned}$$

Comparing (5.1) and (3.17) we have $\phi(-q, 0, 0, q) = f(-q, 0, -q, q, q)$.

By (4.23), (4.25), (4.26), and (5.2) we have *

$$(5.4)\quad \begin{aligned}\phi(0,0,-qx,q) &= 1 + \Sigma q^{\nu^2}(-q)'_\nu x^\nu \\ &= (qx)\{1 + \Sigma(-)^\nu q^{3\nu^2}(-q^2;q^2)'_\nu (qx)'_{2\nu} x^{2\nu}\} \\ &= (q^2x;q^2)\{1 + \Sigma q^{\nu^2}(-q^2;q^2)'_\nu (q^2x;q^2)'_\nu x^\nu\} \\ &= (qx;q^2)\{1 + \Sigma q^{\nu^2+\nu}(-q^2;q^2)'_\nu\ (qx;q^2)'_\nu x^\nu\} \\ &= (-qx)'\{1 + \Sigma(-)^\nu q^{\nu(5\nu-1)/2}(1-q^{2\nu}x)\,(-qx)_{\nu-1}(-q)'_\nu x^{2\nu}\}.\end{aligned}$$

Again by (4.24), (5.1), and by symmetry in x we have

$$(5.5)\quad \begin{aligned}\phi(0,0,q^3x^2,q^2) &= (qx)\{1 + \Sigma q^{\nu(\nu+1)/2}(-q)'_\nu (qx)'_\nu (-x)^\nu\} \\ &= (-q^2x^2;q^2)\{1 + \Sigma q^{\nu(3\nu+1)/2}\ (-q)'_\nu (-q^2x^2;q^2)'_\nu x^{2\nu}\} \\ &= (-qx)\{1 + \Sigma q^{\nu(\nu+1)/2}(-q)'_\nu (-qx)'_\nu x^\nu\}.\end{aligned}$$

Similarly we find

$$(5.6)\quad \begin{aligned}\phi(0,0,x^2,q^4) &= (-qx;q^2)\{1 + \Sigma q^{\nu^2}(-q^2;q^2)'_\nu\,(-qx;q^2)'_\nu x^\nu\} \\ &= (-q^2x^2;q^4)\{1 + \Sigma q^{3\nu^2-\nu}(-q^2;q^2)'_\nu (-q^2x^2;q^4)'_\nu x^{2\nu}\} \\ &= (qx;q^2)\{1 + \Sigma q^{\nu^2}(-q^2;q^2)'_\nu\ (qx;q^2)'_\nu (-x)^\nu\}.\end{aligned}$$

By (5.4) and (4.17) we get

$$(5.7)\quad \begin{aligned}\phi(0,0,-q,q) &= 1 + \Sigma q^{\nu^2}(-q)'_\nu = (q)\{1 + \Sigma(-)^\nu q^{3\nu^2}(-q^2;q^2)'_\nu (q)'_{2\nu}\} \\ &= (q^2;q^2)\{1 + \Sigma q^{\nu^2}(-q^4;q^4)'_\nu\} = (q;q^2)\{1 + \Sigma q^{\nu^2+\nu}(-q^2;q^2)'_\nu (q;q^2)'_\nu\} \\ &= (-q)'\{1 + \Sigma(-)^\nu q^{\nu(5\nu-1)/2}(1+q^\nu)\} = 1/\Pi\,(1-q^{5\nu-4})\,(1-q^{5\nu-1}).\end{aligned}$$

Similarly we find

$$(5.8)\quad \begin{aligned}\phi(0,0,-q^4,q^4)/\Pi\,(1+q^{4\nu+2}) &= 1 + \Sigma(-)^\nu q^{3\nu^2+3\nu}(-q^2;q^2)'_\nu (q^6;q^4)'_\nu \\ &= 1/\Pi\,(1-q^{20\nu-16})\,(1-q^{20\nu-4})\,(1+q^{4\nu+2}).\end{aligned}$$

By (5.6) we have two other expressions for the same function. Again by (5.5) and (4.17) we have

$$(5.9)\quad \begin{aligned}\phi(0,0,-q^2,q^2)/\Pi\,(1+q^{2\nu-1}) &= 1 + \Sigma(-)^\nu q^{\nu(3\nu-1)/2}(-q)'_\nu (q;q^2)'_\nu \\ &= 1/\Pi\,(1-q^{10\nu-8})\,(1-q^{10\nu-2})\,(1+q^{2\nu-1}),\end{aligned}$$

and by (5.5) two other members may be added. In all we can obtain fourteen different representations of $\phi(0, 0, -q, q)$.

We could also write a similar set of identities for the function $\phi(0, 0, -q^2x, q)$, which for $x = 1$ is the function appearing in Ramanujan's second theorem.† For their beauty in themselves we state the following:

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* Cf. L. J. Rogers, *Proceedings of the London Mathematical Society*, Vol. 25 (1894), p. 330.

† This theorem is stated by MacMahon, *loc. cit.*, p. 36.

$$(5.10)\quad \phi(0,0,q^3,q^2)=(q)\{1+\Sigma(-)^\nu q^{\nu(\nu+1)/2}(-q^2;q^2)'_\nu\}$$
$$=(-q)\{1+\Sigma q^{\nu(\nu+1)/2}[(-q)'_\nu]^2\}$$
$$=(-q^2;q^2)\{1+\Sigma q^{\nu(3\nu+1)/2}(-q)'_\nu(-q^2;q^2)'_\nu\},$$
$$(5.11)\quad \phi(0,0,1,q^4)=(-q;q^2)\{1+\Sigma q^{\nu^2}(-q)'_{2\nu}\}$$
$$=(q;q^2)\{1+\Sigma(-)^\nu q^{\nu^2}(-q^2;q^2)'_\nu(q;q^2)'_\nu\}$$
$$=(-q^2;q^4)\{1+\Sigma q^{3\nu^2-\nu}(-q^2;q^2)'_\nu(-q^2;q^4)'_\nu\}.$$

By (4.23) we get

$$(5.12)\quad \phi(\beta x,0,\delta x,q)=1+\Sigma q^{\nu(\nu-1)/2}(\beta,\delta)_\nu(-q)'_\nu(-x)^\nu$$
$$=(-\beta x)\{1+\Sigma q^{\nu^2-\nu}(-q)'_\nu(-\beta x)'_\nu(-\delta x)^\nu\}$$
$$=(-\beta x)(\delta\beta^{-1})\{1+\Sigma(-\delta/\beta)^\nu(-q)'_\nu(-\beta x)'_\nu\},$$
$$(5.13)\quad \phi(q,0,-q^2,q)=1+\Sigma(-)^\nu q^{\nu(\nu+1)/2}=(-q)\{1+\Sigma q^{\nu^2+\nu}[(-q)'_\nu]^2\},$$
$$(5.14)\quad \phi(-q,0,q^2,q)=1+\Sigma q^{\nu(\nu+1)/2}=(q)\{1+\Sigma(-)^\nu q^{\nu^2+\nu}(-q^2;q^2)'_\nu\},$$

and this by (3.14) $=f(-q,-1,q^2,0,q^2)$. Thus we have a new expression for this function. From (5.12), (5.13), and (5.14) we get

$$(5.15)\quad \phi(q,0,-q^2,q)+\phi(-q,0,q^2,q)=2\phi(-q^3,0,q^7,q^4),$$
$$\phi(-q,0,q^2,q)-\phi(q,0,-q^2,q)=2q\phi(-q^5,0,q^9,q^4).$$

Again we have by (5.12)

$$(5.16)\quad \phi(q,0,q^2,q)=1+\Sigma(-)^\nu q^{\nu(\nu+1)/2}(q)_\nu(-q)'_\nu$$
$$=(-q)\{1+\Sigma(-)^\nu q^{\nu^2+\nu}[(-q)'_\nu]^2\},$$
$$(5.17)\quad \phi(-q,0,q^3,q)=1+\Sigma q^{\nu(\nu+1)/2}(1-q^{\nu+1})/(1-q)=1/(1-q).$$

By (4.23)

$$(5.18)\quad \phi(qx,0,qx,q)=1+2\Sigma q^{\nu(\nu+1)/2}(q)_{\nu-1}(-q)'_\nu(-x)^\nu$$
$$=(-qx)\{1+\Sigma q^{\nu^2}(-q)'_\nu(-qx)'_\nu(-x)^\nu\},$$
$$(5.19)\quad \phi(-q,0,-q,q)=1+2\Sigma q^{\nu(\nu+1)/2}(q)_{\nu-1}(-q)'_\nu$$
$$=(q)\{1+\Sigma q^{\nu^2}(-q^2;q^2)'_\nu\},$$

which by (3.11)

$$=(q)f(0,-1,-q,0,q^2)=f(-q,-1,-q,0,q^2).$$

Note the two expressions for the latter as given by (3.2). Making use of expressions for $f(0,-1,-q,0,q^2)$ as given in § 3, we obtain several other members for (5.19). From (5.1) we have

$$(5.20)\quad \phi(qx,-1,0,q)=1+\Sigma q^{\nu(\nu+1)/2}(-q^2;q^2)'_\nu(-x)^\nu$$
$$=(-qx)\{1+\Sigma q^{\nu(3\nu+1)/2}(-q^2;q^2)'_\nu(-qx)'_\nu x^\nu\}$$
$$=(-qx)(q)'\{1+\Sigma q^{\nu(\nu+1)/2}(-q)'_\nu(-qx)'_\nu\}.$$

By comparison with (5.7), (3.1), and (5.3) we easily see that

$$(5.21)\quad \phi(-q,-1,0,q^2)=\phi(0,0,-q,q)/\Pi(1+q^{2\nu}) \\ =1/\Pi(1+q^{2\nu})(1-q^{5\nu-4})(1-q^{5\nu-1}),$$

$$(5.22)\quad \phi(q,-1,0,q)=f(q,-1,q,0,q)\phi(-q,1,0,q).$$

From (5.20) we may show

$$(5.23)\quad \phi(-q,-1,0,q)+\phi(q,-1,0,q) \\ =2(q^3;q^4)(-q^2;q^4)'\phi(q^2,-q^{-1},0,q^4)=2\phi(-q^3,q^{-2},0,q^4),$$

and by comparing with (5.1)

$$(5.24)\quad \phi(-q,-1,0,q)-\phi(q,-1,0,q) \\ =2q(q;q^4)(-q^2;q^4)'\phi(q^6,-q^{-3},0,q^4) \\ =[2q/(1-q^2)]\phi(-q,q^2,0,q^4).$$

We point out two other identities of interest,

$$(5.25)\quad \phi(0,1,q^3x,q)=1+\Sigma q^{\nu(\nu+5)/2}[(-q)'_\nu]^2(-x)^\nu \\ =(qx)\{1+\Sigma q^{\nu(\nu+1)/2}(-q;q^2)_\nu[(-q)'_\nu]^2(qx;q^2)'_\nu\}$$

$$(5.26)\quad \phi(x,\gamma,-qx,q)=1+\Sigma q^{\nu(\nu-1)/2}(-\gamma q)'_\nu(-x)^\nu \\ =(-x)\{1+\Sigma q^{\nu^2}x^\nu(-q)'_\nu(-x)'_\nu(1-\gamma)/(1-\gamma q^\nu)\}.$$

By comparison with § 2 we have

$$(5.27)\quad \phi(-qx,\gamma,q^2x,q)=f(0,-\gamma,-qx,0,q),$$

and by comparison with (5.1) we have

$$(5.28)\quad \phi(x,1,-qx,q)=(-x)\phi(0,q^{-1}x,0,q)=\phi(x,0,0,q).$$

Again comparing with § 2 (5.1), and (3.10),

$$(5.29)\quad \phi(x,-1,-qx,q)=f(0,1,x,0,q),$$

$$(5.30)\quad \phi(-1,-1,q,q)=2(q)\phi(q,0,0,q^2)=2\phi(-q,0,q,q) \\ =2(q)f(q,-1,0,0,q)=f(0,1,-1,0,q).$$

Another way of obtaining identities from those already found is illustrated by the following. On equating coefficients of γ^k in the last two members of (5.26) and using (3.8) we get

$$(5.31)\quad \Sigma q^{[\nu(\nu-1)/2]+k}(-q^\nu)_k(-q)'_k(-x)^\nu \\ =-(-x)\{\Sigma q^{\nu^2+(k-1)\nu}(-q)'_{\nu-1}(-x)'_\nu x^\nu\}.$$

Replacing x by -1, dividing by q^k, and reducing, this becomes

$$(5.32)\quad 1+\Sigma q^{\nu(\nu+1)/2}(-q^{\nu+1})_k(-q)'_k=(q)\{1+\Sigma(-)^\nu q^{\nu^2+(k+1)\nu}(-q^2;q^2)_\nu\} \\ =(q)\phi(q^{k+2},0,0,q^2)=\phi(-q,0,q^{k+2},q),$$

the last two members being obtained by comparison with (5.1) and (5.12). Then we may write

$$(5.33)\quad \phi(-1,\gamma,q,q)=\phi(-1,0,q,q)+\Sigma_k\phi(-q,0,q^{k+2},q)q^k\gamma^k.$$

But by (5.26) and (5.12) we have

$$\phi(-1,0,q,q)=(q)\{1+\Sigma(-)^{\nu}q^{\nu^2}(-q^2;q^2)'_{\nu}+1+\Sigma(-)^{\nu}q^{\nu^2+\nu}(-q^2;q^2)'_{\nu}\}$$
$$=(q)\{(-q;q^2)+(-q^2;q^2)\}=1+(q)(-q^2;q^2)=1+(-q^2;q^2)(-q;q^2)'.$$

By (5.12) and (3.1)

$$\phi(-q,0,q^{k+2},q)=(-q)'_k f(-q^{k+1},-1,q^2,0,q^2)$$
$$=(-q^2;q^2)(-q^{k+1};q^2)'(-q)'_k.$$

Then (5.33) becomes

$$(5.34)\quad \phi(-1,\gamma,q,q)=1+(-q^2;q^2)(-q;q^2)'$$
$$+\Sigma_k(-q^2;q^2)(-q^{k+1};q^2)(-q)'_k(\gamma q)^k$$
$$=1+(-q^2;q^2)(-q;q^2)'$$
$$\times\{1+\Sigma_k(-q^2;q^2)_k(\gamma q)^{2k}\}+\Sigma_k(-q;q^2)_k(\gamma q)^{2k-1}$$
$$=f(-q,-\gamma^2,0,q-\gamma q,q^2)$$
$$+(-q^2;q^2)(-q;q^2)'\{1+\Sigma_k(-q^2;q^2)_k(\gamma q)^{2k}\}.$$

Then by (3.10)

$$(5.35)\quad \phi(-1,\gamma,q,q)-f(-q,-\gamma^2,0,q-\gamma q,q^2)$$
$$=(-q^2;q^2)(-q;q^2)'(-q^2\gamma^2;q^2)',$$

which obviously implies an identity of several members.

By (4.23) we get

$$(5.36)\quad \phi(x,\gamma,-q^2x,q)=1+\Sigma q^{\nu(\nu-1)/2}(-\gamma q)'_{\nu}(-x)^{\nu}(1-q^{\nu+1})/(1-q)$$
$$=(-x)\{1+\Sigma q^{\nu^2+\nu}(-q^{-1}\gamma)_{\nu}(-\gamma q)'_{\nu}(-x)'_{\nu}x^{\nu}\}$$
$$=(-x)(-q^2)(-\gamma q)'\{1+\Sigma q^{2\nu}(-\gamma q^{-1})_{\nu}(-q)'_{\nu}(-x)'_{\nu}\},$$

which with (5.26) gives the identity

$$(5.37)\quad \phi(q^3,q^2,-q^5,q)=(-q^3)\phi(q^5,q^4,-q^7,q^2);$$

and

$$(5.38)\quad \phi(qx,\gamma,q^2x,q)=1+\Sigma q^{\nu(\nu+1)/2}(q)_{\nu}(-q)'_{\nu}(-\gamma q)'_{\nu}(-x)^{\nu}$$
$$=(-qx)\{1+\Sigma q^{\nu^2+\nu}(\gamma)_{\nu}(-q)'_{\nu}(-\gamma q)'_{\nu}(-qx)'_{\nu}(-x)^{\nu}$$
$$=(qx)(q)(-q\gamma)'\{1+\Sigma(-qx)^{\nu}(\gamma)_{\nu}(-q)'_{\nu}(-qx)'_{\nu}\}$$

which with previous results gives

$$(5.39)\quad \phi(x,-x,qx,q)=f(qx,-1,x,0,q).$$

Other relations of this kind might be found. From such relations we could evidently state identities of many members as we have already done for a number of identities involving known functions.

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SUMMATION FACTORS WHICH ARE POWERS OF A COMPLEX VARIABLE.

By WALTER H. DURFEE.

1. *Introduction.* By a well known theorem of Frobenius,* if the power series $y(z) = \sum_{n=0}^{\infty} a_n z^n$ has the unit circle as circle of convergence, and if $\sum_{n=0}^{\infty} a_n$ is summable by Cesàro's k-th mean to the value s, then $\lim y(z) = s$ as $z \to +1$ along any path lying wholly between two fixed chords of the unit circle intersecting at $z = +1$.

Frobenius stated the theorem for the case, $k = 1$. Extensions have been made by various writers, notably Hölder † and Fejér.‡ Hurwitz,§ also, has shown that if $\sum_{n=0}^{\infty} a_n$ is summable C_k to the value s, and if $g_n(z)$ is a function of z such that, 1) $\lim_{z \to 1} g_n(z) = 1$, 2) for each $|z| < 1$, $n^k g_n(z)$ is bounded for all n,¶ and 3) $\sum_{n=0}^{\infty} n^k |\Delta^{k+1} g_n(z)|$ converges for each $|z| < 1$ and is bounded for all $|z| < 1$; then $\sum_{n=0}^{\infty} a_n g_n(z)$ converges for $|z| < 1$, and $\lim_{z \to 1} \sum_{n=0}^{\infty} a_n g_n(z) = s$.

The theorems of this paper deal with functions which satisfy conditions equivalent to these, the series under discussion being of the form $y(z) = \sum_{n=0}^{\infty} a_n z^{f(n)}$, where a_n is real and z is a complex variable. The inquiry is directed toward determining the conditions on $f(n)$ under which the series approaches a limit as $z \to +1$.

For the purposes of this paper the functions $f(n)$ are restricted to the class of logarithmico-exponential functions,‖ so expressed that $f(0) = 0$; for brevity these will be designated as L-functions. Certain properties of these

* *Journal für die reine und angewandte Mathematik*, Vol. 89 (1880), p. 262.

† *Mathematische Annalen*, Vol. 20 (1882), p. 535.

‡ *Mathematische Annalen*, Vol. 58 (1904), p. 62.

§ Cf. Morse, "Certain Definitions of Summability," *American Journal of Mathematics*, Vol. 45 (1923), p. 263.

¶ The bound may depend upon z.

‖ Hardy, *Orders of Infinity*, III, 2.

functions, which will be useful in the subsequent pages, are: if $f(x)$ is an L-function, then $f(x)$ and all its derivatives are ultimately continuous, non-vanishing, monotonic, and of constant sign; as $x \to +\infty$, $f(x)$ approaches either infinity, or zero, or some other definite limit; $f'(x)$ is also an L-function, and $f''(x) = o[f'(x)]$ entails $f'(x) = o[f(x)]$.

Fejér, in his paper on Fourier Series,* proved that Frobenius' theorem is valid for series of the form $y(z) = \sum_{n=0}^{\infty} a_n z^{n^p}$, where p may be any integer greater than zero. It has been shown, however, by Hardy, in a paper "On Certain Oscillating Series," † that if $f(n)$ has too rapid a rate of increase the series $\sum_{n=0}^{\infty} a_n z^{f(n)}$ may not approach any limit as $z \to +1$. A discussion of certain series of this type will be found on a later page.

2. *Statement of Theorems.* The discussion will be facilitated by the definition of a region within the unit circle, which we shall call R, to which the variable z is confined. As in Frobenius' theorem, let there be two chords intersecting at the point $z = +1$. They may, without loss of generality, be considered as making equal angles with the real axis. We now define R as the open region, in a neighborhood of $z = +1$, bounded by these chords and interior to them.

THEOREM I. *If the series* $\sum_{n=0}^{\infty} a_n$ *is summable* C_k *to the value* s, *and if* a) $f(x)$ *is an* L*-function, and* $f(0) = 0$, b) $\log x = o[f(x)]$, c) $f(x) > 0$, $f'(x) > 0$, and d) $f''(x)/f'(x) = O[1/x]$, *then the series* $y(z) = \sum_{n=0}^{\infty} a_n z^{f(n)}$ *converges for* $|z| < 1$, *and* $\lim y(z) = s$ *as* $z \to 1$ *along any path lying wholly in* R.‡

THEOREM II. *If* $\sum_{n=0}^{\infty} a_n$ *is summable* C_1 *to the value* s, *and if* a) $f(x)$ *is an* L*-function, and* $f(0) = 0$, b) $f(x) = O[\log x]$, c) $f(x) \to +\infty$, $f'(x) > 0$, and d) $a_n = O[\lambda^{f(n)} f'(n)]$, or $s_n = O[\lambda^{f(n)}]$, *for every* $\lambda > 1$, *then the series* $y(z) = \sum_{n=0}^{\infty} a_n z^{f(n)}$ *converges for* $|z| < 1$, *and* $\lim y(z) = s$ *as* $z \to 1$ *along any path lying wholly in* R.

3. *Proof of Lemmas.* It will be convenient at this point to establish

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* *Mathematische Annalen*, Vol. 58 (1904), p. 62.

† *Quarterly Journal of Mathematics*, Vol. 38 (1906-7), p. 269.

‡ A somewhat similar theorem, due to H. L. Garabedian, has recently been published. See *Annals of Mathematics*, Vol. 32 (1931), p. 83.

the truth of a number of lemmas which will be of use in the proof of the above theorems.

LEMMA 1. *As $z \to 1$ in R, $|\log z| = O(\log \rho)$, where $\rho = |z|$.*

Let θ be the amplitude of z, $-\pi/2 < \theta < \pi/2$. Then $|\log z| \leqq |\log \rho| + |\theta|$. As $z \to 1$, $\theta \sim \sin\theta$, and since $\log\rho = \log[1-(1-\rho)] \sim -(1-\rho)$, $|\log\rho| \sim 1-\rho$. Now let $|z-1| = r$, and call the acute angle between $(z-1)$ and the real axis β. Then $\sin\theta = r/\rho \sin\beta$; and $\rho^2 = r^2 + 1 - 2r\cos\beta$. Noting that as $z \to 1$, $(1+\rho)/\rho \to 2$, and $2\cos\beta - r \to 2\cos\beta$, we have

$$\frac{|\theta|}{|\log\rho|} \sim \frac{\sin\theta}{1-\rho} = \frac{(1+\rho)\sin\theta}{1-\rho^2}$$

$$= \frac{[(1+\rho)r/\rho]\sin\beta}{r(2\cos\beta - r)} = \frac{(1+\rho)\sin\beta}{\rho(2\cos\beta - r)} \sim \frac{2\sin\beta}{2\cos\beta} = \tan\beta.$$

Since z lies in R, $\tan\beta$ is less than some constant, A, determined by the chords bounding R, whence

$$|\log z| \leqq |\log\rho| + A|\log\rho| = O(\log\rho).$$

LEMMA 2. *If $f(x)$ is an L-function, and if $f(x)$ and its first k derivatives exist and are continuous, non-vanishing, and of constant sign for $x \geqq a$, then for such values of x they either all bear the same sign, or all derivatives up to a certain one bear the same sign and thereafter they alternate.*

To prove this, suppose that for $x \geqq a$, $g(x) \equiv f^{(i)}(x)$ and $g'(x)$ are of opposite sign. By Taylor's Theorem, for $x \geqq a$,

$$g(x) = g(a) + (x-a)g'(a) + [(x-a)^2/2]\, g''\{a + \theta(x-a)\} \qquad 0 < \theta < 1.$$

Then, for $x > a$ we may write,

$$g''\{a + \theta(x-a)\} = [2/(x-a)^2]\{g(x) - (x-a)g'(a) - g(a)\}.$$

If $g(a) > 0$, and $g'(a) < 0$, then by hypothesis $g(x) > 0$ for $x > a$, and the first two terms in the braces on the right-hand side of the above expression are, signs considered, positive; while the last, which is negative, is a constant. The second term increases indefinitely with x; hence for sufficiently great values of x, $g''\{a + \theta(x-a)\} > 0$. But for $x > a$, $a + \theta(x-a) > a$, that is, there are values of $x > a$ for which $g''(x) > 0$. Since by hypothesis $g''(x)$ is of constant sign for $x \geqq a$, it follows that for all $x \geqq a$, $g''(x) > 0$.

If, on the other hand, $g(a) < 0$, and $g'(a) > 0$, then for $x > a$, $g(x) < 0$,

and for sufficiently great values of x, $g''\{a+\theta(x-a)\}<0$. It follows that there are values of $x>a$ for which $g''(x)<0$, and hence that $g''(x)<0$ for all $x \geqq a$.

Thus if $g(x)$ and $g'(x)$ are of opposite sign for $x \geqq a$, then $g(x)$ and $g''(x)$ are of like sign for $x \geqq a$; which proves the lemma.

LEMMA 3. *If* $f''(x)/f'(x) = O(1/x)$, *then for any* $k \geqq 1$, $f^{(k)}(x)/f'(x) = O(1/x^{k-1})$.

The statement is obviously true for $k=1$. Suppose it true for some $k>1$. Differentiating, we have

$$\frac{f'f^{(k+1)} - f^{(k)}f''}{(f')^2} = \frac{f^{(k+1)}}{f'} - \frac{f^{(k)}}{f'} \cdot \frac{f''}{f'}$$

$$= \frac{f^{(k+1)}}{f'} - O\left(\frac{1}{x^k}\right) = O\left[\frac{d}{dx}\left(\frac{1}{x^{k-1}}\right)\right] = O\left[\frac{1}{x^k}\right],$$

or $f^{(k+1)}/f' = O[1/x^k]$. Thus, if the statement is true for $k=2$, as by hypothesis it is, then it is true for all $k \geqq 2$, and the proof is complete.

LEMMA 4. *If* $f(x) = O(\log x)$, *and if* $f(x) \to +\infty$ *and* $f'(x)$ *is positive, then* $f''(x)$ *is negative for sufficiently great values of* x; *and further,* $f''(x)/f'(x) = O(1/x)$.

If $f(x) = O(\log x)$, then $f'(x) = O(1/x)$*; that is, for sufficiently large values of x, $|f'(x)| < M/x$, where M is some positive constant. Hence $f'(x) \to 0$ as $x \to +\infty$. Since $f(x)$ is an L-function, so is $f'(x)$; and since $f'(x) \to 0$, ultimately monotonically, $f''(x)$ must be ultimately negative. The truth of the first statement in the lemma is thus established. The function $xf''(x)/f'(x)$ is an L-function, and therefore approaches either zero, or $-\mu$, or $-\infty$, where μ is some positive constant. Suppose that, for $x \geqq X$, $xf''(x)/f'(x) < -\alpha$, where $\alpha > 1$. Then $f''(x)/f'(x) + \alpha/x < 0$. Integrating from X to x, we have

$$\log f'(x) + \alpha \log x - \log f'(X) - \alpha \log X < 0, \quad \text{or} \quad x^\alpha f'(x) < q(X),$$

where $$q(X) = X^\alpha f'(X).$$

Putting this in the form $f'(x) - q(X)/x^\alpha < 0$, and integrating once more from X to x,

$$f(x) - q(X)/(1-\alpha)x^{\alpha-1} - f(X) + q(X)/(1-\alpha)X^{\alpha-1} < 0,$$

or $$f(x) < f(X) + q(X)/(\alpha-1)X^{\alpha-1} - q(X)/(\alpha-1)x^{\alpha-1}.$$

* Hardy, *Orders of Infinity*.

Then we have $\limsup_{x\to+\infty} f(x) \leqq f(X) + Xf'(X)/(\alpha-1)$, which contradicts the hypothesis $f(x)\to+\infty$. It follows that $\lim_{x\to+\infty} xf''(x)/f'(x) = -\mu$ exists, and $-1 \leqq -\mu \leqq 0$, whence also $f''(x)/f'(x) = O(1/x)$.

LEMMA 5. *The terms in the k-th derivative, with respect to n, of $z^{f(n)}$ can be set in one-to-one correspondence with the partitions of k, the components of any partition being the orders of the derivatives of $f(n)$ in the corresponding term, so that the number of occurrences of a component in a partition indicates the exponent of that derivative. The number of components of a partition is the same as the exponent of $\log z$ in the corresponding term. The numerical coefficients of the terms are all positive.*

If we set $\phi(n) = z^{f(n)}$, the first three derivatives are

$$\phi'(n) = z^{f(n)}f'(n)\log z,$$
$$\phi''(n) = z^{f(n)}\{(f'(n)\log z)^2 + f''(n)\log z\},$$
$$\phi'''(n) = z^{f(n)}\{(f'(n)\log z)^3 + 3f'(n)f''(n)(\log z)^2 + f'''(n)\log z\}.$$

Inspection shows that the statements in the lemma are valid for $k \leqq 3$. Suppose now that they are true for some fixed k. Then each term of $\phi^{(k)}$ is of the form $T_\phi = Az^f(f')^{a_1}(f'')^{a_2}\cdots(f^{(k)})^{a_k}(\log z)^h$, where $h = \sum_{i=1}^{k} a_i$, any, or all but one, of the a_i may be zero, and $k = \sum_{i=1}^{k} ia_i$.

From any partition of k partitions of $(k+1)$ may be formed, either by adding a new component, 1, or by increasing any single component by unity; and if each partition of k be so treated in every possible way all the partitions of $(k+1)$ will be obtained.

Now, $\phi^{(k+1)} = d\phi^{(k)}/dn$, and by differentiating T_ϕ we obtain

$$T'_\phi = dT_\phi/dn = Az^f\{(f')^{a_1+1}(f'')^{a_2}\cdots(f^{(k)})^{a_k}(\log z)^{h+1} + (f')^{a_1}\cdots(f^{(k)})^{a_k}\sum_{i=1}^{k} a_i[f^{(i+1)}/f^{(i)}](\log z)^h\}.$$

These terms correspond to the partitions of $(k+1)$ obtainable from that partition of k which corresponds to T_ϕ. Further, in each term of T'_ϕ the sum of the exponents is equal to the exponent of $\log z$, and $\sum ia_i = k+1$. Finally, the numerical coefficients are either A or Aa_i, and in either case they bear the same sign as A.

Thus if the statements of the lemma are true for some particular k, by induction they are also true for $(k+1)$. Being verified for $k \leqq 3$, they hold for every positive integral value of k.

4. *Proof of Theorems.* We proceed now to the consideration of the first Theorem. In showing that, under the conditions of this theorem $y(z)$ converges within the unit circle it will be advantageous to go further and prove that for $|z|<1$, $\sum_{n=0}^{\infty}|a_n z^{f(n)}|$ is convergent. We note that since $\sum_{n=0}^{\infty} a_n$ is C_k-summable, $|a_n|<Gn^k$, where G is a positive constant; it will therefore suffice to prove the convergence of $\sum_{n=0}^{\infty} n^k\rho^{f(n)}$, $0\leqq\rho<1$, $\log n = o[f(n)]$. Since $f(n)/\log n\to+\infty$, for a fixed ρ, and $n>N$, we have $[f(n)/\log n]\log\rho<-k-2$. That is, $f(n)\log\rho<-(k+2)\log n$, or $\rho^{f(n)}<1/n^{k+2}$. Then $n^k\rho^{f(n)}<1/n^2$ for $n>N$. The convergence of $\sum_{n=0}^{\infty} n^k\rho^{f(n)}$ thus follows from that of $\sum_{n=0}^{\infty} 1/n^2$, and the absolute convergence of $y(z)$ within the unit circle is thus established.

The series $\sum_{n=0}^{\infty} a_n$ being summable C_k, we have $\lim_{n\to\infty} C_n^{(k)}=s$, where

$$C_n^{(k)}=S_n^{(k)}/\binom{n+k}{k}=[S_0^{(k-1)}+S_1^{(k-1)}+\cdots+S_n^{(k-1)}]/\binom{n+k}{k},$$

and $S_n^{(0)}=s_n=a_0+a_1+\cdots+a_n$. Thus, if we define $C_{-r}^{(k)}=0$ for $r>0$, we have

$$S_n^{(k-1)}=S_n^{(k)}-S_{n-1}^{(k)}=\binom{n+k}{k}C_n^{(k)}-\binom{n+k-1}{k}C_{n-1}^{(k)}$$
$$S_n^{(k-2)}=S_n^{(k-1)}-S_{n-1}^{(k-1)}=\binom{n+k}{k}C_n^{(k)}-2\binom{n+k-1}{k}C_{n-1}^{(k)}+\binom{n+k-2}{k}C_{n-2}^{(k)}.$$

and so on. This process ultimately yields

$$S_n^{(0)}=s_n=\binom{n+k}{k}C_n^{(k)}-\binom{k}{1}\binom{n+k-1}{k}C_{n-1}^{(k)}$$
$$+\binom{k}{2}\binom{n+k-2}{k}C_{n-2}^{(k)}-+\cdots+(-1)^k\binom{k}{k}\binom{n}{n}C_{n-k}^{(k)},$$
$$a_n=s_n-s_{n-1}=\binom{n+k}{k}C_n^{(k)}-\binom{k+1}{1}\binom{n+k-1}{k}C_{n-1}^{(k)}$$
$$+\cdots+(-1)^k\binom{k+1}{k}\binom{n}{k}C_{n-k}^{(k)}+(-1)^{k+1}\binom{k+1}{k+1}\binom{n-1}{k}C_{n-k-1}^{(k)}.$$

This may be written,

$$a_n=\sum_{p=0}^{k+1}(-1)^p\binom{k+1}{p}\binom{n+k-p}{k}C_{n-p}^{(k)}$$

and accordingly we have

$$y(z)=\sum_{n=0}^{\infty}\sum_{p=0}^{k+1}(-1)^p\binom{k+1}{p}\binom{n+k-p}{k}C_{n-p}^{(k)}z^{f(n)}.$$

If now we set up the expression.

$$\binom{k+1}{0}\left\{\binom{k}{k}C_0^{(k)}+\binom{1+k}{k}C_1^{(k)}z^{f(1)}+\binom{2+k}{k}C_2^{(k)}z^{f(2)}+\cdots\right\}$$
$$-\binom{k+1}{1}\left\{\binom{k-1}{k}C_{-1}^{(k)}+\binom{k}{k}C_0^{(k)}z^{f(1)}+\binom{1+k}{k}C_1^{(k)}z^{f(2)}+\cdots\right\}$$
$$+\binom{k+1}{2}\left\{\binom{k-2}{k}C_{-2}^{(k)}+\binom{-k-1}{k}C_{-1}^{(k)}z^{f(1)}+\binom{k}{k}C_0^{(k)}z^{f(2)}+\cdots\right\}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+(-1)^{k+1}\binom{k+1}{k+1}\left\{\binom{k-k-1}{k}C_{-k+1}^{(k)}+\binom{k-k}{k}C_{-k}^{(k)}z^{f(1)}+\binom{k-k+1}{k}C_{-k+1}^{(k)}z^{f(2)}+\cdots\right.$$

each row, in view of the definition $C_{-r}^{(k)}=0$, $r>0$, is seen to be a series of the form $\sum_{n=0}^{\infty}\binom{k+1}{q}\binom{n+k}{k}C_n^{(k)}z^{f(n+q)}$. The terms of this series are not greater in absolute value than the corresponding terms of $\sum_{n=0}^{\infty}\binom{k+1}{q}(n+1)^kC_n^{(k)}z^{f(n)}$, for $f(n+q)\geqq f(n)$, and

$$\binom{n+k}{k}=\frac{(n+k)(n+k-1)\cdots(n+2)(n+1)}{k(k-1)\cdots2\cdot1}\leqq(n+1)^k.$$

Since also $C_n^{(k)}$ approaches a definite limit as $n\to\infty$, this latter series converges with $\sum_{n=0}^{\infty}n^kz^{f(n)}$, the convergence of which has already been established. It follows that each of the row-series in the above expression is convergent, $|z|<1$. Since the number of rows, $(k+2)$, is finite, the series formed of the sums of columns also converges and represents the same function as the sum of the row-series. But the series of column-sums is exactly the function,

$$y(z)=\sum_{n=0}^{\infty}\sum_{p=0}^{k+1}(-1)^p\binom{k+1}{p}\binom{n+k-p}{k}C_{n-p}^{(k)}z^{f(n)}.$$

It thus appears that we can write

$$y(z)=\binom{k+1}{0}\sum_{n=0}^{\infty}\binom{n+k}{k}C_n^{(k)}z^{f(n)}-\binom{k+1}{1}\sum_{n=0}^{\infty}\binom{n+k-1}{k}C_{n-1}^{(k)}z^{f(n)}$$
$$+-\cdots+(-1)^{k+1}\binom{k+1}{k+1}\sum_{n=0}^{\infty}\binom{n-1}{k}C_{n-k-1}^{(k)}z^{f(n)}.$$

Rewriting this in such manner as to omit terms containing $C_{-r}^{(k)}$, $r>0$, we have

$$y(z)=\binom{k+1}{0}\sum_{n=0}^{\infty}\binom{n+k}{k}C_n^{(k)}z^{f(n)}-\binom{k+1}{1}\sum_{n=0}^{\infty}\binom{n+k}{k}C_n^{(k)}z^{f(n+1)}$$
$$+-\cdots+(-1)^{k+1}\binom{k+1}{k+1}\sum_{n=0}^{\infty}\binom{n+k}{k}C_n^{(k)}z^{f(n+k+1)},$$

or

$$y(z)=\sum_{n=0}^{\infty}\binom{n+k}{k}\left\{\sum_{p=0}^{k+1}(-1)^p\binom{k+1}{p}z^{f(n+p)}\right\}C_n^{(k)}=\sum_{n=0}^{\infty}b_n(z)C_n^{(k)},$$

where $$b_n(z) = \binom{n+k}{k} \sum_{p=0}^{k+1} (-1)^p \binom{k+1}{p} z^{f(n+p)}.$$

It must now be shown that the transformation whose coefficients are $b_n(z)$ is regular; that is, that the limit of the sum of the series $\sum_{n=0}^{\infty} b_n(z) C_n^{(k)}$ as $z \to 1$ in R is the same as the limit, s, of the sequence $C_n^{(k)}$. The necessary and sufficient conditions for regularity in this case are four,* namely,

I. For each n, $\lim_{z \to 1} b_n(z) = 0$.

II. For each z in R, $\sum_{n-0}^{\infty} | b_n(z) |$ converges.

III. For all z in R, $\sum_{n-0}^{\infty} | b_n(z) |$ is bounded.

IV. $\lim_{z \to 1} \sum_{n-0}^{\infty} b_n(z) = 1$.

Since

$$\lim_{z \to 1} b_n(z) = \binom{n+k}{k} \sum_{p=0}^{k+1} (-1)^p \binom{k+1}{p} = \binom{n+k}{k} (1-1)^{k+1} = 0,$$

for each value of n, condition I is evidently satisfied.

Bearing in mind that $\binom{n+k}{k} \leqq (n+1)^k$, and that $| z^{f(n+p)} | \leqq \rho^{f(n)}$ for $p \geqq 0$, we have

$$| b_n(z) | \leqq \binom{n+k}{k} \sum_{p=0}^{k+1} \binom{k+1}{p} \rho^{f(n)} \leqq 2^{k+1} (n+1)^k \rho^{f(n)}.$$

Thus

$$\sum_{n-0}^{\infty} | b_n(z) | \leqq 2^{k+1} \sum_{n-0}^{\infty} (n+1)^k \rho^{f(n)},$$

which converges with $\sum_{n-0}^{\infty} n^k \rho^{f(n)}$. Therefore the series $\sum_{n-0}^{\infty} | b_n(z) |$ is convergent for each z in R, and condition II is satisfied.

It is equally simple to deal with condition IV. We have

$$C_n^{(k)} = \left[1 / \binom{n+k}{k}\right] \left\{ \binom{n+k}{k} a_0 + \binom{n+k-1}{k} a_1 + \cdots + \binom{k}{k} a_n \right\}.$$

In particular, if $a_0 = 1$ and $a_i = 0$, $i > 0$, then for every n, $C_n^{(k)} = a_0 = 1$. In this case, therefore,

$$y(z) = \sum_{n=0}^{\infty} a_n z^{f(n)} = a_0 z^{f(0)} = 1,$$

and likewise

$$y(z) = \sum_{n=0}^{\infty} b_n(z) C_n^{(k)} = \sum_{n=0}^{\infty} b_n(z).$$

* *Bulletin of the American Mathematical Society*, Vol. 28 (1922), p. 17.

Equating these expressions for $y(z)$ we see that $\sum_{n=0}^{\infty} b_n(z) = 1$, and accordingly condition IV is satisfied.

There remains condition III. We note, in the first place, that since $f(x)$ is an L-function there is a value of x, say n_0, such that for $x > n_0$, $f(x)$ and its first $(k+1)$ derivatives exist, and are continuous, monotonic, and of constant sign. If, therefore, we set $\phi(n) = z^{f(n)}$, we have, for $n > n_0$, the first difference

$$\Delta\phi(n) = \phi(n) - \phi(n+1) = \int_{n+1}^{n} \phi'(r_1)\, dr_1,$$

the second difference

$$\Delta^2\phi(n) = \Delta\phi(n) - \Delta\phi(n+1) = \int_{n+1}^{n}\int_{r_1+1}^{r_1} \phi''(r_2)\, dr_2 dr_1,$$

and the $(k+1)$-st difference

$$\Delta^{k+1}\phi(n) = \Delta^k\phi(n) - \Delta^k\phi(n+1) = \int_{n+1}^{n}\int_{r_1+1}^{r_1}\cdots\int_{r_k+1}^{r_k} \phi^{(k+1)}(r_{k+1})\, dr_{k+1}\cdots dr_2 dr_1.$$

Now, in this last expression, set $r_1 = n + s_1$, $r_2 = r_1 + s_2$, and in general $r_i = r_{i-1} + s_i$.

$$\Delta^{k+1}\phi(n) = \int_1^0\int_1^0\cdots\int_1^0 \phi^{(k+1)}(\mu)\, ds_{k+1}\cdots ds_2 ds_1,$$

where $\mu = n + s_1 + s_2 + \cdots + s_{k+1}$. Observing that

$$| b_n(z) | = \binom{n+k}{k} \Big| \sum_{p=0}^{k+1} (-1)^p \binom{k+1}{p} z^{f(n+p)} \Big| \leqq (n+1)^k \, | \Delta^{k+1}\phi(n) |,$$

and that the above expression for $\Delta^{k+1}\phi(n)$ is valid for $n > n_0$, we may now write

$$\begin{aligned}\sum_{n=0}^{\infty} | b_n(z) | &\leqq \sum_{n=0}^{n_0} | b_n(z) | \\ &\quad + \sum_{n=n_0+1}^{\infty} (n+1)^k \Big| \int_1^0\int_1^0\cdots\int_1^0 \phi^{(k+1)}(\mu)\, ds_{k+1}\cdots ds_2 ds_1 \Big| \\ &\leqq \sum_{n=0}^{n_0} | b_n(z) | \\ &\quad + \sum_{n=n_0+1}^{\infty} (n+1)^k \int_0^1\int_0^1\cdots\int_0^1 | \phi^{(k+1)}(\mu) |\, ds_{k+1}\cdots ds_2 ds_1.\end{aligned}$$

Since $| b_n(z) | \leqq 2^{k+1}(n+1)^k \rho^{f(n)}$, and $\rho^{f(n)} < 1$, the first summation is less than $2^{k+1}[1 + 2^k + \cdots + (n_0+1)^k]$, and is certainly bounded for all z in R. We wish then to show that

$$\sum_{n=n_0+1}^{\infty}(n+1)^k\int_0^1\int_0^1\cdots\int_0^1 |\phi^{k+1}(\mu)|\,ds_{k+1}\cdots ds_2 ds_1$$

is bounded for all z in R.

It is evident that the statements made in Lemma 5 regarding $z^{f(n)}$ are equally valid for $\rho^{f(n)}$, where $\rho = |z|$. If we set $\psi(n) = \rho^{f(n)}$, and indicate by T_ψ a term of $\psi^{(k)}(n)$,

$$T_\psi = A\rho^f (f')^{\alpha_1}(f'')^{\alpha_2}\cdots(f^{(k)})^{\alpha_k}(\log\rho)^h, \qquad h=\sum_{i=1}^{k}\alpha_i,$$

we note that $T'_\psi = T_\psi\{f'\log\rho + \sum_{i=1}^{k}\alpha_i f^{(i+1)}/f^{(i)}\}$.

Suppose now that the alternation of signs of the $f^{(i)}$ commences with f''. Then, for $n > n_0$, $f^{(i)}$ and $f^{(i+1)}$ have opposite signs, and since $f'\log\rho < 0$, it follows that T_ψ and T'_ψ bear opposite signs, and hence also $\psi^{(k)}$ and $\psi^{(k+1)}$. On observing that $\psi'(n) = \rho^{f(n)}f'(n)\log\rho < 0$, we conclude that

$$(-1)^{k+1}\psi^{(k+1)}(n) > 0, \qquad n > n_0.$$

By Lemma 1, and the conclusion just stated,

$$|\phi^{(k+1)}(n)| \leqq B\,|\psi^{(k+1)}(n)| = (-1)^{k+1}B\psi^{(k+1)}(n),$$

where z is in R and B is some positive constant. Since $\mu \geqq n$, we can write

$$\sum_{n_0+1}^{\infty}(n+1)^k\int_0^1\int_0^1\cdots\int_0^1 |\phi^{(k+1)}(\mu)|\,ds_{k+1}\cdots ds_2 ds_1$$

$$\leqq B\sum_{n_0+1}^{\infty}\int_0^1\int_0^1\cdots\int_0^1 (-1)^{k+1}(\mu+1)^k\psi^{(k+1)}(\mu)\,ds_{k+1}\cdots ds_2 ds_1.$$

A first integration gives

$$B\sum_{n_0+1}^{\infty}\int_0^1\int_0^1\cdots\int_0^1 (-1)^{k+1}[(\mu+1)^k\psi^{(k)}(\mu) - k(\mu+1)^{k-1}\psi^{(k-1)}(\mu)$$

$$+ k(k-1)(\mu+1)^{k-2}\psi^{(k-2)}(\mu) - +\cdots+(-1)^k k!\,\psi(\mu)]_{s_{k+1}=0}^{s_{k+1}=1}\,ds_k\cdots ds_2 ds_1.$$

This may be expressed as $B\sum_{n_0+1}^{\infty}\{g(n) - g(n+1)\}$, where

$$g(n) = \int_0^1\int_0^1\cdots\int_0^1 (-1)^k\{(\mu_1+1)^k\psi^{(k)}(\mu_1)$$

$$- k(\mu_1+1)^{k-1}\psi^{(k-1)}(\mu_1) + -\cdots+(-1)^k k!\,\psi(\mu_1)\}ds_k\cdots ds_2 ds_1,$$

in which $\mu_1 = n + s_1 + s_2 + \cdots + s_k$.

The terms in the integrand on the right-hand side of this expression are of the form $(-1)^m(k!/m!)(\mu_1+1)^m\psi^{(m)}(\mu_1)$, so that if we set

$$U_m = \int_0^1\int_0^1 \cdots \int_0^1 (-1)^m(k!/m!)(\mu_1+1)^m\psi^{(m)}(\mu_1)\,ds_k \cdots ds_2 ds_1$$

we can write $g(n) = \sum_{m=0}^{k} U_m$. Noting that $\mu_1 \leqq n+k$, we see that

$$U_m \leqq \frac{k!\,(n+k+1)^m}{m!}\int_0^1\int_0^1 \cdots \int_0^1 (-1)^m\psi^{(m)}(\mu_1)\,ds_k \cdots ds_2 ds_1$$

$$= \frac{k!\,(n+k+1)^m}{m!}\int_0^1\int_0^1 \cdots \int_0^1 (-1)^m\left[\psi^{(m-1)}(\mu_1)\right]_{s_k=0}^{s_k=1} ds_{k-1} \cdots ds_2 ds_1.$$

Now if m is even, $\psi^{(m-1)}(\mu_1)$ is negative and increasing, if m is odd, $\psi^{(m-1)}(\mu_1)$ is positive and decreasing; in either case, therefore, $(-1)^m\psi^{(m-1)}(\mu_1)$ is negative and increasing, and hence

$$U_m < \frac{k!\,(n+k+1)^m}{m!}\int_0^1\int_0^1 \cdots \int_0^1 (-1)^{m-1}\psi^{(m-1)}(\mu_2)\,ds_{k-1} \cdots ds_2 ds_1,$$

where $\mu_2 = n + s_1 + s_2 + \cdots + s_{k-1}$.

After m such integrations we find

$$U_m < \frac{k!\,(n+k+1)^m}{m!}\int_0^1\int_0^1 \cdots \int_0^1 \psi(\mu_{m+1})\,ds_{k-m} \cdots ds_2 ds_1,$$

where $\mu_{m+1} = n + s_1 + \cdots + s_{k-m}$. Since $\psi(\mu_{m+1}) = \rho^{f(\mu_{m+1})}$ is a positive, monotonically decreasing, function of μ_{m+1}, it follows that

$$0 \leqq U_m < [k!\,(n+k+1)^m/m!]\rho^{f(n)},$$

and hence that

$$\limsup_{n\to\infty} U_m \leqq \limsup_{n\to\infty} [k!\,(n+k+1)^m/m!\,]\rho^{f(n)}.$$

But this limit is zero provided condition (b) of Theorem I, $\log x = o[f(x)]$ is fulfilled; for then

$$\limsup_{n\to\infty} [k!\,(n+k+1)^m/m!]\rho^{f(n)}$$
$$= \limsup_{n\to\infty} (k!/m!\,)\exp[m\log(n+k+1) + f(n)\log\rho] = 0.$$

Thus $\lim_{n\to\infty} U_m = 0$, and as there are but a finite number, $(k+1)$, of such terms in $g(n)$, we have $\lim_{n\to\infty} g(n) = 0$. As a result,

$$B\sum_{n_0+1}^{\infty}\{g(n)-g(n+1)\} = Bg(n_0+1)$$
$$< B\sum_{m=0}^{k}[k!\,(n_0+k+2)^m/m!\,]\rho^{f(n_0+1)} \leqq Bk!\sum_{m=0}^{k}[n_0+k+2)^m/m!\,],$$

which is bounded.

Collecting these results, we have

$$\sum_{n-0}^{\infty}|\,b_n(z)\,| < 2^{k+1}[1+2^k+\cdots+(n_0+1)^k]+Bk!\sum_{m=0}^{k}(n_0+k+2)^m/m!.$$

It is thus proved that providing the alternation of signs of the derivatives of $f(x)$ commences with $f''(x)$ condition III is satisfied, $\sum_{n-0}^{\infty}|\,b_n(z)\,|$ is bounded for all z in R.

On the other hand, if $f''(x) > 0$, then for this, and possibly other derivatives, $\operatorname{sgn} f^{(i)} = (-1)^i$. These may enter certain terms of $\phi^{(k+1)}(n)$, for example $T_{\phi_1}, T_{\phi_2}, \cdots, T_{\phi_q}$, in such manner as to make the signs of these terms different from that of the leading term, $z^{f(n)}\{f'(n)\log z\}^{k+1}$. In this case we have

$$|\,\phi^{(k+1)}(n)\,| \leqq (-1)^{k+1}B\psi^{(k+1)}(n)+(-1)^k 2B(T_{\psi_1}+T_{\psi_2}+\cdots+T_{\psi_q}).$$

The series $\sum_{n_0+1}^{\infty}(n+1)^k\,|\,\Delta^{k+1}\phi(n)\,|$ is bounded if $\sum_{n_0+1}^{\infty} n^k\,|\,\Delta^{k+1}\phi(n)\,|$ is bounded; for the sake of simplicity the proof will be given for the latter. We have

$$\sum_{n_0+1}^{\infty} n^k\,|\,\Delta^{k+1}\phi(n)\,| \leqq B\sum_{n_0+1}^{\infty}\int_0^1\int_0^1\cdots\int_0^1 (-1)^{k+1}\mu^k\psi^{(k+1)}(\mu)\,ds_{k+1}\cdots ds_2 ds_1$$
$$+2B\sum_{n_0+1}^{\infty} n^k\int_0^1\int_0^1\cdots\int_0^1 (-1)^k(T_{\psi_1}+T_{\psi_2}+\cdots+T_{\psi_q})\,ds_{k+1}\cdots ds_2 ds_1.$$

The first of these summations, after a first integration, gives as before, $B\sum_{n_0+1}^{\infty}\{g(n)-g(n+1)\}$, where now

$$g(n) = \int_0^1\int_0^1\cdots\int_0^1 (-1)^k\{\mu_1{}^k\psi^{(k)}(\mu_1) - k\mu_1{}^{k-1}\psi^{(k-1)}(\mu_1)$$
$$+-\cdots+(-1)^k k!\,\psi(\mu_1)\}\,ds_k\cdots ds_2 ds_1$$

and $\mu_1 = n+s_1+s_2+\cdots+s_k$. Here again we may put $g(n) = \sum_{m=0}^{k} U_m$,

where,

$$U_m = \int_0^1 \int_0^1 \cdots \int_0^1 (-1)^m (k!/m!) \mu_1{}^m \psi^{(m)}(\mu_1) ds_k \cdots ds_2 ds_1.$$

In this case, however, we cannot be sure that $(-1)^m \psi^{(m-1)}(\mu_1)$ is negative and increasing; so the former procedure cannot be followed beyond this point.

But the terms of $\psi^{(m)}(\mu_1)$ are all of the form

$$T_\psi = A\rho^f (f')^{\alpha_1} (f'')^{\alpha_2} \cdots (f^{(m)})^{\alpha_m} (\log \rho)^h, \qquad h = \Sigma\alpha_i, \quad m = \Sigma i\alpha_i.$$

By Lemma 3, and condition (d) of the theorem, there is a positive constant, M_i, such that, for n sufficiently great,

$$| f^{(i)}(\mu_1) |^{\alpha_i} < M_i{}^{\alpha_i} [f'(\mu_1)]^{\alpha_i} / \mu_1{}^{\alpha_i(i-1)}.$$

Supposing n_0 so chosen that this inequality holds for all values of i occurring in any of the $T\psi$, and setting $\prod_{i=1}^{m} M_i{}^{\alpha_i} = M$, we have

$$| T_\psi | < A\rho^{f(\mu_1)} M [f'(\mu_1)]^h | \log \rho |^h / \mu_1{}^{m-h}.$$

Thus the absolute values of the terms in the integrand,

$$(-1)^m (k!/m!) \mu_1{}^m \psi^{(m)}(\mu_1)$$

are less than

$$(AMk!/m!) \rho^{f(\mu_1)} \mu_1{}^h [f'(\mu_1)]^h,$$

provided $\rho > 1/e$. In this expression A, M, k, m, are constants $\mu_1 \leqq n + k$, $f'(\mu_1) \leqq f'(n + k)$, and $\rho^{f(\mu_1)} \leqq \rho^{f(n)}$; hence

$$\limsup_{n\to\infty} \rho^{f(\mu_1)} \mu_1{}^h [f'(\mu_1)]^h \leqq \limsup_{n\to\infty} \exp\{f(n)\log\rho + h[\log(n+k) + \log f'(n+k)]\}.$$

But $\log n = o[f(n)]$, and since $f'' = o(f')$ entails $f' = o(f)$, it follows that the limit on the right-hand side of the above expression is zero.

There being a finite number of such terms in the integrand of the expression for U_m, it is clear that $\lim_{n\to\infty} U_m = 0$, and hence that $\lim_{n\to\infty} g(n) = 0$. We thus have

$$B \sum_{n_0+1}^{\infty} \{g(n) - g(n+1)\} = Bg(n_0 + 1).$$

But

$$| g(n_0+1) | \leqq \sum_{m=0}^{k} | U_m(n_0+1) |$$

$$\leqq \sum_{m=0}^{k} (k!/m!) \int_0^1 \int_0^1 \cdots \int_0^1 \sum_{T\psi} \mu_1{}^m | T\psi | ds_k \cdots ds_2 ds_1,$$

where the summation in the integrand extends over all the terms in $\psi^{(m)}$, and $\mu_1 = n_0 + 1 + s_1 + \cdots + s_k$. It has also been shown that

$$| T_\psi | < A\rho^{f(\mu_1)} M[f'(\mu_1)]^h | \log \rho |^h / {\mu_1}^{m-h} < AM[f'(\mu_1)]^h / {\mu_1}^{m-h} \quad \text{if} \quad \rho > 1/e.$$

Hence

$$| g(n_0+1) | < \sum_{m=0}^{k} AM(k!/m!) \sum_h \int_0^1 \int_0^1 \cdots \int_0^1 {\mu_1}^h [f'(\mu_1)]^h ds_k \cdots ds_2 ds_1;$$

and since $\mu_1 \leqq n_0 + k + 1$ and $f'(\mu_1)$ is a positive, monotonically increasing function of μ_1,

$$| g(n_0+1) | < \sum_{m=0}^{k} AM(k!/m!) \sum_h (n_0+k+1)^h [f'(n_0+k+1)]^h,$$

where h takes on all values corresponding to the terms T_ψ in $\psi^{(m)}$.

Thus $g(n_0+1)$ is bounded for all z in R, and so also is the first summation in the expression for $\sum_{n_0+1}^{\infty} n^k \mid \Delta^{k+1}\phi(n) \mid$.

With regard to the second of these summations, we note that any of the T_{ψ_i} therein is of the form

$$T_{\psi_i} = A_i \rho^f (f')^{a_1} (f'')^{a_2} \cdots (f^{(k+1)})^{a_{k+1}} (\log \rho)^h, \qquad h = \sum_{i=1}^{k+1} a_i.$$

Suppose now that the derivatives for which $\operatorname{sgn} f^{(i)} = (-1)^i$ are $f^{(i_1)}$, $f^{(i_2)}$, $\cdots$, $f^{(i_r)}$, and for these let

$$a_{i_1} + a_{i_2} + \cdots + a_{i_r} = \alpha, \qquad i_1 a_{i_1} + i_2 a_{i_2} + \cdots + i_r a_{i_r} = \beta.$$

By Lemma 3, and condition (d), we may set

$$| f^{(i)}(\mu) |^{a_i} < {M_i}^{a_i} [f'(\mu)]^{a_i} / \mu^{a_i(i-1)},$$

and accordingly

$$| T_{\psi_i} | < A_i M \rho^f (f')^{a_1+\alpha} \cdots (f^{(k+1)})^{a_{k+1}} | \log \rho |^h / \mu^{\beta-\alpha},$$

where now the $f^{(i_1)}, \cdots, f^{(i_r)}$, no longer appear, and $\sum a_i = h$, but $\sum i a_i = k - \beta + \alpha + 1 = j + 1 < k + 1$. Then we have

$$\sum_{n_0+1}^{\infty} n^k \int_0^1 \int_0^1 \cdots \int_0^1 (-1)^k T_{\psi_i} ds_{k+1} \cdots ds_2 ds_1$$

$$< A_i M \sum_{n_0+1}^{\infty} \int_0^1 \int_0^1 \cdots \int_0^1 (-1)^l \mu^j \rho^f (f')^{a_1+\alpha} \cdots (f^{(k+1)})^{a_{k+1}} (\log \rho)^h ds_{k+1} \cdots ds_2 ds_1$$

But the integrand in the latter expression is, except for a constant factor,

one of the terms of $(-1)^{j+1}\mu^j\psi^{(j+1)}(\mu)$. If the terms in $\psi^{(j+1)}$ are all of like sign,

$$\sum_{n_0+1}^{\infty} n^k \int_0^1\int_0^1 \cdots \int_0^1 (-1)^k T\psi_\iota ds_{k+1}\cdots ds_2 ds_1$$

$$< G\sum_{n_0+1}^{\infty}\int_0^1\int_0^1\cdots\int_0^1 (-1)^{j+1}\mu^j\psi^{(j+1)}(\mu)\,ds_{k+1}\cdots ds_2 ds_1,$$

which can be shown bounded as was done for $\psi^{(k+1)}$ in like circumstances. But if the terms are not all of like sign the above process is repeated with $\psi^{(j+1)}$ instead of $\psi^{(k+1)}$. As the order of the derivative of ψ is lowered at each repetition we are certain ultimately to reach the form

$$\sum_{n_0+1}^{\infty}\int_0^1\int_0^1\cdots\int_0^1 -\psi'(\mu)\,ds_{k+1}\cdots ds_2 ds_1,$$

which can readily be proved bounded.

The second summation being thus proved bounded, it is established that $\sum_{n=0}^{\infty} |\, b_n(z)\,|$ is bounded for all z in R; condition III is satisfied. This completes the proof of the regularity of the transformation whose coefficients are $b_n(z)$. It follows that, as $z\to 1$ in R,

$$\lim_{z\to 1} y(z) = \lim_{z\to 1}\sum_{n=0}^{\infty} b_n(z)C_n^{(k)} = \lim_{n\to\infty} C_n^{(k)} = s,$$

and the proof of Theorem I is complete.

Owing to the less general nature of Theorem II, its proof is considerably simpler. We set up the series

$$w(z) = \sum_{n=0}^{\infty}(n+1)\{z^{f(n)} - 2z^{f(n+1)} + z^{f(n+2)}\}C_n^{(1)} = \sum_{n=0}^{\infty} b_n(z)C_n^{(1)};$$

and the proof consists in showing that, under the conditions of the theorem, $w(z)\to s$ as $z\to 1$ in R, and that for all z in R, $y(z) = \sum_{n=0}^{\infty} a_n z^{f(n)}$ converges to the same value as $w(z)$.

As before, since $\lim_{n\to\infty} C_n^{(1)} = s$, we must establish the regularity of the transformation whose coefficients are $b_n(z)$. The conditions for regularity are those given in the proof of Theorem I.

It is evident that for each n, $\lim_{x\to 1} b_n(z) = (n+1)(1-2+1) = 0$, so that condition I is satisfied.

Since $f(x) = O(\log x)$, by Lemma 4 the alternation of signs of the derivatives begins with $f''(x)$. Accordingly, if we again set $\phi(n) = z^{f(n)}$, $\psi(n) = \rho^{f(n)}$, and recall Lemma 1, we can find an n_0 such that for $n > n_0$, $|\phi^{(k)}(n)| \leqq B|\psi^{(k)}(n)| = (-1)^k B\psi^{(k)}(n)$, where z is in R and B is some positive constant. We also have, as before, but this time with $k = 1$,

$$|b_n(z)| = (n+1)|z^{f(n)} - 2z^{f(n+1)} + z^{f(n+2)}| = (n+1)|\Delta^2\phi(n)|.$$

Thus

$$\sum_{n=0}^{\infty} |b_n(z)| = \sum_{n=0}^{n_0} |b_n(z)| + \sum_{n=n_0+1}^{\infty} (n+1)|\Delta^2\phi(n)|.$$

Since the absolute value of $b_n(z)$ is not greater than $4(n+1)$, the first summation is less than $4\{1+2+\cdots+(n_0+1)\} = 2(n_0+1)(n_0+2)$. With regard to the second summation it will suffice to consider $\sum_{n=n_0+1}^{\infty} n|\Delta^2\phi(n)|$.

We have $\Delta^2\phi(n) = \int_1^0\int_1^0 \phi''(\mu)\,ds_2 ds_1$, where $\mu = n + s_1 + s_2$, and so

$$\sum_{n=n_0+1}^{\infty} n|\Delta^2\phi(n)| = \sum_{n=n_0+1}^{\infty} n\left|\int_1^0\int_1^0 \phi''(\mu)\,ds_2 ds_1\right|$$

$$\leqq \sum_{n=n_0+1}^{\infty} n\int_0^1\int_0^1 |\phi''(\mu)|\,ds_2 ds_1 \leqq B\sum_{n_0+1}^{\infty} n\int_0^1\int_0^1 \psi''(\mu)\,ds_2 ds_1.$$

Now, $\psi''(\mu) = \rho^{f(\mu)}\{[f'(\mu)\log\rho]^2 + f''(\mu)\log\rho\}$, and by Lemma 4 there can be found constants, $M_1 > 0$ and $M_2 > 0$, such that for $n > n_0$, $f'(x) < M_1/x$, $|f''(x)| < M_2 f'(x)/x$. Finally, for $\rho > 1/e$, $|\log\rho| > [\log\rho]^2$; whence

$$\psi''(\mu) < -\rho^{f(\mu)}(\log\rho)\{M_1 f'(\mu)/\mu + M_2 f'(\mu)/\mu\} = -M\rho^{f(\mu)}[f'(\mu)/\mu](\log\rho),$$

where $M = M_1 + M_2$. Then, since $n \leqq \mu$

$$\sum_{n_0+1}^{\infty} n|\Delta^2\phi(n)| < BM\sum_{n_0+1}^{\infty} n\int_0^1\int_0^1 -\rho^{f(\mu)}[f'(\mu)/\mu](\log\rho)\,ds_2 ds_1$$

$$\leqq BM\sum_{n_0+1}^{\infty}\int_0^1\int_0^1 -\rho^{f(\mu)}f'(\mu)(\log\rho)\,ds_2 ds_1$$

$$= BM\sum_{n_0+1}^{\infty}\int_0^1 \left[-\rho^{f(\mu)}\right]_{s_2=0}^{s_2=1} ds_1 = BM\sum_{n_0+1}^{\infty}\{g(n) - g(n+1)\},$$

where $g(n)=\int_0^1 \rho^{f(n+s_1)}ds_1 < \rho^{f(n)}$. Therefore $\limsup_{n\to\infty} g(n) \leqq \lim_{n\to\infty} \rho^{f(n)} = 0$, that is to say, $\lim_{n\to\infty} g(n) = 0$. On the other hand, $g(n_0+1) < \rho^{f(n_0+1)} < 1$, whence $\sum_{n_0+1}^{\infty} n \mid \Delta^2\phi(n) \mid < BM$, which is bounded.

We have thus proved that for all z in R, $\sum_{n=0}^{\infty} \mid b_n(z) \mid$ is bounded; since the series is one of positive terms it is also convergent for each z in R. Conditions II and III are thus satisfied.

To test the series for IV, $\lim_{z\to 1} \sum_{n=0}^{\infty} b_n(z) = 1$, we suppose as before that $a_0 = 1$, $a_i = 0$, $i > 0$, whence for each n, $C_n^{(1)} = a_0 = 1$. Thus, in this case, $w(z) = \sum_{n=0}^{\infty} b_n(z)$, and $y(z) = \sum_{n=0}^{\infty} a_n z^{f(n)} = a_0 z^{f(0)} = 1$. If now it can be shown that, for all z in R, $w(z) = y(z)$, then IV will certainly be satisfied.

To this end we note that $a_n = (n+1)C_n^{(1)} - 2nC_{n-1}^{(1)} + (n-1)C_{n-2}^{(1)}$, so that

$$y(z) = \sum_{n=0}^{\infty} \{(n+1)C_n^{(1)} - 2nC_{n-1}^{(1)} + (n-1)C_{n-2}^{(1)}\} z^{f(n)}.$$

If we indicate by y_m and w_m the sum of the first $(m+1)$ terms of $y(z)$ and $w(z)$ respectively, we have

$$\begin{aligned} y_n - w_{n-2} &= \{(n+1)C_n^{(1)} - 2nC_{n-1}^{(1)}\} z^{f(n)} + nC_{n-1}^{(1)} z^{f(n-1)} \\ &= \{(n+1)C_n^{(1)} - nC_{n-1}^{(1)}\} z^{f(n)} + \{z^{f(n-1)} - z^{f(n)}\} nC_{n-1}^{(1)}. \end{aligned}$$

The coefficient in the first term of this expression is simply s_n, and by condition (d) of the theorem for a given $\lambda > 1$ there is a constant H such that for all n sufficiently great $\mid s_n \mid \leqq H\lambda^{f(n)}$. If, therefore, for a fixed value of z, we choose λ so that $1/\lambda > \mid z \mid = \rho$, then for n sufficiently great $\mid s_n z^{f(n)} \mid = \mid s_n \mid \rho^{f(n)} \leqq H(\lambda\rho)^{f(n)}$. But the value of this quantity approaches zero as n increases, wherefore the first term in the above expression for $(y_n - w_{n-2})$ also approaches zero as n increases.

The second term is $n[\Delta z^{f(n-1)}]C_{n-1}^{(1)}$, and the absolute value of this is less than $Dn \mid C_{n-1}^{(1)} \mid \int_0^1 -\rho^{f(\mu)} f'(\mu)(\log\rho)ds$, where $\mu = n-1+s$, and D is a positive constant. Since $[f'(\mu)] < M_1/\mu \leqq M_1/(n-1)$, and $\mid \log\rho \mid < 1$ if $\rho > 1/e$, and $\int_0^1 \rho^{f(\mu)}ds < \rho^{f(n-1)}$, we can say that

$$n \mid [\Delta z^{f(n-1)}]C_{n-1}^{(1)} \mid < DM_1 n/(n-1) \mid C_{n-1}^{(1)} \mid \rho^{f(n-1)};$$

and this expression certainly approaches zero as n increases. As a result we

have $\lim_{n\to\infty}(y_n - w_{n-2}) = 0$, or $y(z) = w(z)$ for all z in R. Condition IV is accordingly satisfied, and the transformation is regular; hence as $z \to 1$ in R, $\lim_{z\to 1} y(z) = \lim_{z\to 1} w(z) = \lim_{n\to\infty} C_n^{(1)} = s$.

It remains only to show that $y(z)$ is convergent everywhere within the unit circle. Its convergence within the region R has been established, and it is obvious that for any fixed z, $|z| < 1$, we may so choose the chords defining R that z will lie within the angle formed by them. We now define R' as that part of the annular region between circles of radius $\epsilon > 0$ and $(1-\epsilon)$, with centers at the origin, which lies within this same angle, ϵ being so chosen that z lies in R'.

Recalling the proof for the case where z was confined to R, we have in the present case also for n greater than some n_0, $|\phi^{(k)}(n)| \leqq (-1)^k B\psi^{(k)}(n)$, and $\sum_{n=0}^{\infty} |b_n(z)| = \sum_{n=0}^{n_0} |b_n(z)| + \sum_{n_0+1}^{\infty} (n+1)|\Delta^2\phi(n)|$. The first summation is bounded, and the second likewise provided that $B\sum_{n_0+1}^{\infty} n\int_0^1\int_0^1 \psi''(\mu)\,ds_2 ds_1$ is bounded. Since $|\log\rho| < |\log\epsilon|$ we now have

$$\psi''(\mu) < -M|\log\epsilon|\rho^{f(\mu)}[f'(\mu)/\mu](\log\rho)$$

and hence $B\sum_{n_0+1}^{\infty} n\int_0^1\int_0^1 \psi''(\mu)\,ds_2 ds_1 < BM|\log\epsilon|$. It follows that for z in R', $\sum_{n=0}^{\infty}|b_n(z)|$, and hence $w(z)$, is bounded and convergent.

Again,

$$y_n - w_{n-2} = \{(n+1)C_n^{(1)} - nC_{n-1}^{(1)}\}z^{f(n)} + \{z^{f(n-1)} - z^{f(n)}\}nC_{n-1}^{(1)}.$$

The first term approaches zero as n increases, and for the second

$$n|\{z^{f(n-1)} - z^{f(n)}\}C_{n-1}^{(1)}| = n|[\Delta z^{f(n-1)}]C_{n-1}^{(1)}| < DM_1 n/(n-1)|C_{n-1}^{(1)}\log\epsilon)|\rho^{f(n-1)},$$

which also approaches zero as n increases. It thus appears that $y(z) = w(z)$, and $y(z)$ consequently converges, for all z in R'.

It is clear that for any fixed z, $0 < |z| < 1$, ϵ may be so chosen that z will lie within R'. It is obvious also that $y(z)$ is convergent for $z = 0$. It thus follows that $y(z)$ is convergent everywhere within the unit circle; and this completes the proof of Theorem II.

The proof as given makes use of the second form of condition (d),

$s_n = O[\lambda^{f(n)}]$ for every $\lambda > 1$. The condition $a_n = O[\lambda^{f(n)} f'(n)]$ for every $\lambda > 1$, is stronger than $s_n = O[\lambda^{f(n)}]$; it is sufficient to assure the fulfillment of the latter. For, if the former holds, then there is a constant H, and an N, such that for $n > N$, $a_n < H\lambda^{f(n)} f'(n)(\log \lambda)$. Thus for $n > N$,

$$|s_n| \leqq |s_N| + |a_{N+1}| + \cdots + |a_n| < |s_N| + H(\log \lambda)\{\lambda^{f(N+1)} f'(N+1) \cdots + \lambda^{f(n)} f'(n)\}.$$

Now λ can be so chosen that $\lambda^{f(n)} f'(n)$ is ultimately a decreasing function; for since $f(n) = O(\log n)$ and $f'(n) = O(1/n)$, an N_1 can be found such that for $n > N_1$, $f(n) < M_3(\log n)$ and $f'(n) < M_1/n$; hence for such values of n, $\lambda^{f(n)} f'(n) < \lambda^{M_3(\log n)} \cdot M_1/n = M_1 e^{(M_3 \log \lambda - 1)\log n}$. Since M_3 does not depend upon the choice of λ the latter may be chosen so small that $(M_3 \log \lambda - 1) < 0$, whence $M_1 e^{(M_3 \log \lambda - 1)\log n} \to 0$. Accordingly, for $n > N$, N_1, we can write

$$|s_n| < |s_N| + H \int_N^n \lambda^{f(x)} f'(x)(\log \lambda)\, dx = |s_N| + H\lambda^{f(n)} - H\lambda^{f(N)}.$$

Then $$\frac{|s_n|}{\lambda^{f(n)}} < \frac{|s_N| - H\lambda^{f(N)}}{\lambda^{f(n)}} + H, \quad \text{and} \quad \limsup_{n\to\infty} \frac{|s_n|}{\lambda^{f(n)}} \leqq H.$$

But it follows that an N_2 can be found such that for $n > N_2$, $|s_n| \leqq H\lambda^{f(n)}$, or, what is the same thing, $s_n = O[\lambda^{f(n)}]$.

Either form of condition (d) is sufficient. On the other hand, $a_n = o[\lambda^{f(n)}]$ is necessary. For if $\sum_{n=0}^{\infty} a_n x^{f(n)}$, $0 < x < 1$, is convergent, then $\sum_{n=0}^{\infty} a_n/\lambda^{f(n)}$ is also convergent for all $\lambda > 1$. It follows that $\lim_{n\to\infty} a_n/\lambda^{f(n)} = 0$, whence $a_n = o[\lambda^{f(n)}]$.

5. *Oscillating Series.* It was mentioned in the introductory paragraphs that if the exponents of z in the series $\sum_{n=0}^{\infty} a_n z^{f(n)}$ have too rapid a rate of increase the series does not necessarily approach any limit as $z \to 1$. Hardy* cites as particular instances the series $\sum_{n=0}^{\infty} (-1)^n x^{p^n}$, $p \geqq 2$ integer, and $\sum_{n=0}^{\infty} (-1)^n x^{n!}$; he shows that these both oscillate about the value $\frac{1}{2}$, the Cesàro mean of their coefficients, as $x \to 1^-$ along the real axis.

Belinfante† has proved that the power-series $y(x) = \sum_{n=0}^{\infty} (-1)^n x^{f(n)}$,

* *Quarterly Journal of Mathematics*, Vol. 38 (1906-7), p. 269.

† *Koninklijle Akademie van Wetenschappen te Amsterdam*, Verslagen 32 (1923), p. 472.

$f(n)$ integer, approaches no limit as $x \to 1^-$, provided two constants, $a > 1$ and b, can be found such that for all sufficiently great n, $1 + (a^2 - 1)/2a < b \leqq f(n+1)/f(n) \leqq a$. The function $f(n) = p^n$, $p > 1$ integer, falls into this category, for then we may choose $a = b = p$, and $1 + (p^2 - 1)/2p < p = p^{n+1}/p^n$.

This theorem may be extended to the case where $f(n)$ is not an integer as follows. Let $[f(n)]$ be the greatest integer in $f(n)$, so that $[f(n)] \leqq f(n) < [f(n)+1]$, and

$$\frac{[f(n+1)]}{[f(n)+1]} < \frac{f(n+1)}{f(n)} < \frac{[f(n+1)+1]}{[f(n)]},$$

whence

$$\frac{f(n)[f(n+1)]}{f(n+1)[f(n)+1]} < 1 < \frac{f(n)[f(n+1)+1]}{f(n+1)[f(n)]}.$$

Then, for all sufficiently great n.

$$1 + \frac{a^2-1}{2a} < b \leqq \frac{f(n+1)}{f(n)} < \frac{f(n+1)}{f(n)} \cdot \frac{f(n)[f(n+1)+1]}{f(n+1)[f(n)]} \leqq \frac{f(n)[f(n+1)+1]}{f(n+1)[f(n)]} a,$$

and likewise,

$$\frac{f(n)[f(n+1)]}{f(n+1)[f(n)+1]} b \leqq \frac{f(n+1)}{f(n)} \cdot \frac{f(n)[f(n+1)]}{f(n+1)[f(n)+1]} < \frac{f(n+1)}{f(n)} \leqq a.$$

Now choose $a_1 > a$, and $b_1 < b$, such that $1 + (a_1^2 - 1)/2a_1 < b_1 < b$.*
Then, since

$$\lim_{n\to\infty} \frac{f(n)[f(n+1)]}{f(n+1)[f(n)+1]} = \lim_{n\to\infty} \frac{f(n)[f(n+1)+1]}{f(n+1)[f(n)]} = 1,$$

we have for all sufficiently great n,

$$1 + \frac{a_1^2-1}{2a_1} < b_1 < \frac{f(n)[f(n+1)]}{f(n+1)[f(n)+1]} b \leqq \frac{[f(n+1)]}{[f(n)+1]} < \frac{f(n+1)}{f(n)} < \frac{[f(n+1)+1]}{[f(n)]} \leqq \frac{f(n)[f(n+1)+1]}{f(n+1)[f(n)]} a < a_1.$$

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* That such a choice can be made is apparent from the fact that $1 + (a^2-1)/2a$ is an increasing function of a. Since $1 + (a^2-1)/2a < b$ an a_1 and b_1 can always be found such that $1 + (a^2-1)/2a < 1 + (a_1^2-1)/2a_1 < b_1 < b$. But then certainly $a_1 < a$.

It now follows that the series with integral exponents,

$$g(x)=x-x^{[f(1)]}+x^{[f(2)+1]}-x^{[f(3)]}+x^{[f(4)+1]}-+\cdots,$$
$$h(x)=x-x^{[f(1)+1]}+x^{[f(2)]}-x^{[f(3)+1]}+x^{[f(4)]}-+\cdots,$$

satisfy the conditions of Belinfante's theorem, and hence approach no limit as $x\to 1^-$. But $[f(2n+1)]\leqq f(2n+1)<[f(2n+1)+1]$, and $f[(2n)+1]>f(2n)\geqq[f(2n)]$, whence $g(x)<y(x)<h(x)$. Now

$$h(x)-g(x)=(1-x)\{x^{[f(1)]}+x^{[f(2)]}+x^{[f(3)]}+\cdots\}.$$

Since $1<b\leqq f(n+1)/f(n)$, ultimately $f(n)\geqq Cb^n$ where C is a positive constant; and hence for any positive constant M, ultimately $[f(n)]>Mn$. Thus there is an n_0 such that for $n\geqq n_0$, $x^{[f(n)]}<x^{Mn}$. Then

$$h(x)-g(x)<(1-x)\{n_0-1+\sum_{n=n_0}^{\infty}x^{[f(n)]}\}<(1-x)\{n_0+\sum_{n=n_0}^{\infty}x^{Mn}\}$$
$$=(1-x)n_0+(1-x)\frac{x^{Mn_0}}{1-x^M}=(1-x)n_0+\frac{x^{Mn_0}}{1+x+\cdots+x^{M-1}}.$$

Hence

$$\limsup_{x\to 1^-}\{h(x)-g(x)\}\leqq\lim_{x\to 1^-}\left\{(1-x)n_0+\frac{x^{Mn_0}}{1+x+\cdots+x^{M-1}}\right\}=\frac{1}{M}.$$

But M was any positive number, whence $\lim_{x\to 1^-}\{h(x)-g(x)\}=0$. It follows that $\lim_{x\to 1^-}\{h(x)-y(x)\}=0$, and $y(x)$ accordingly oscillates between the same limits as $h(x)$.

6. *Limits of Oscillation.* It is interesting to see what can be done toward determining these limits. Hardy has shown that for the series $\sum_{n=0}^{\infty}(-1)^n x^{n!}$ they are actually 0 and 1, and the same is true for any series whose exponents increase at a more rapid rate than $n!$. For the series $y(z)=\sum_{n=0}^{\infty}(-1)^n x^{p^n}$, since $x>y(x)>x-x^p$, it is evident that 0 and 1 constitute lower and upper bounds; but a somewhat closer approximation may be obtained, as follows. Consider the ratio $m(x)=(x^p-x^{p^2})/(x-x^p)$. Its first derivative, with respect to x, is $[(p-1)x^p/(x-x^p)^2]\psi(x^{p-1})$, where $\psi(t)=1-(p+1)t^p+pt^{p+1}$. Now $\psi'(t)=p(p+1)t^{p-1}(t-1)$ is negative if $t<1$, and furthermore, $\psi(1)-\psi(t)=(1-t)\psi'(\eta)<0$, where $t<\eta<1$. Thus for $t<1$, $\psi(t)>\psi(1)=1-(p+1)+p=0$, and since $x^{p-1}<1$, it follows that

$dm(x)/dx > 0$, and $m(x)$ is a function increasing with x. Then $m(x^p)$ increases with x^p, and so also with x, and in general $m(x^{p^k})$ increases with x.

Now suppose that the expression $(x^{p^{2k}} - x^{p^{2k+1}})/(x - x^p)$ increases with x. If so, the product

$$\frac{x^{p^{2k}} - x^{p^{2k+1}}}{x - x^p} \cdot \frac{x^{p^{2k+1}} - x^{p^{2k+2}}}{x^{p^{2k}} - x^{p^{2k+1}}} \cdot \frac{x^{p^{2k+2}} - x^{p^{2k+3}}}{x^{p^{2k+1}} - x^{p^{2k+2}}} = \frac{x^{p^{2k+2}} - x^{p^{2k+3}}}{x - x^p}$$

must also increase with x. But the supposition is true for $k = 1$, since $(x^{p^2} - x^{p^3})/(x - x^p) = m(x) \cdot m(x^p)$ certainly increases with x; hence it is true by induction for any positive integer k. From this we conclude, since

$$\frac{y(x)}{x - x^p} = 1 + \sum_{k=1}^{\infty} \frac{x^{p^{2k}} - x^{p^{2k+1}}}{x - x^p},$$

that $y(x)/(x - x^p)$ is a function increasing with x, and hence that $y(x)/(x - x^p) > y(x^p)/(x^p - x^{p^2})$. Observing now that $y(x^p) = x - y(x)$, we have, on the one hand, $\{(x^p - x^{p^2})/(x - x^p) + 1\} y(x) > x$, or $y(x) > x[(x - x^p)/(x - x^{p^2})]$, and on the other hand, $[(x^p - x^{p^2})/(x - x^p)] \times \{x - y(x^p)\} > y(x^p)$, or $y(x^p) < x\,[(x^p - x^{p^2})/(x - x^{p^2})]$. Taking derivatives of numerator and denominator in these expressions, we find,

$$\lim_{x \to 1^-} \left\{ x \frac{x - x^p}{x - x^{p^2}} \right\} = \lim_{x \to 1^-} \left\{ \frac{1 - px^{p-1}}{1 - p^2 x^{p^2-1}} \right\} = \frac{1}{1+p},$$

and

$$\lim_{x \to 1^-} \left\{ x \frac{x^p - x^{p^2}}{x - x^{p^2}} \right\} = \lim_{x \to 1^-} \left\{ \frac{px^{p-1} - p^2 x^{p^2-1}}{1 - p^2 x^{p^2-1}} \right\} = \frac{p}{1+p}.$$

Thus, $\liminf_{x \to 1^-} y(x) \geqq 1/(1+p)$, and $\limsup_{x \to 1^-} y(x) \leqq p/(1+p)$.

It will be seen that as $p \to 1$ these limits both approach $\frac{1}{2}$. It should be remarked, however, that they do not seem to be very close approximations to the actual limits of oscillation, as is evidenced by numerical calculation for certain particular values of p. The series $y(x)$ converges so slowly for values of x near 1 that this calculation would be very tedious but for a device ascribed by Hardy to Wedderburn. We set up the companion series,

$$g(x) = \frac{1}{2} + \frac{\log x}{1+p} + \frac{1}{2!} \frac{(\log x)^2}{1+p^2} + \frac{1}{3!} \frac{(\log x)^3}{1+p^3} + \cdots,$$

which is convergent for $0 < x$. If we then set $F(x) = y(x) - g(x)$, we have, $F(x) + F(x^p) = y(x) - g(x) + y(x^p) - g(x^p)$

$$= x - \left\{ 1 + \frac{(1+p)\log x}{1+p} + \frac{1}{2!} \frac{(1+p^2)(\log x)^2}{1+p^2} + \cdots \right\} = x - e^{\log x} = 0.$$

Now, $\lim_{x\to 1^-} g(x) = \frac{1}{2}$; therefore, if $y(x)$ oscillates about the value $\frac{1}{2}$ as $x \to 1^-$, $F(x)$ will oscillate about zero with the same amplitude. Further, since $F(x)+F(x^p)=0$, we have $F(x)=F(x^{p^2})=F(x^{p^{2k}})$ for any positive integer k. While $y(x)$ converges slowly for values of x near unity, $g(x)$ converges there very rapidly. A proper choice of k, however, yields series $y(x^{p^{2k}})$ and $g(x^{p^{2k}})$, both of which converge fairly quickly. Thus the relation

$$y(x)=F(x)+g(x)=F(x^{p^{2k}})+g(x)=y(x^{p^{2k}})-g(x^{p^{2k}})+g(x)$$

renders the calculation of $y(x)$ comparatively simple for values of x near unity. In the figure are shown graphs of $F(x)$ for three values of p, namely $p=2.0$, 1.75, and 1.5. It will be noted that the amplitude of the oscillation for $p=2.0$ is of the order of five times that for $p=1.75$, and one hundred times that for $p=1.5$. In the latter case then, the actual range of oscillation *of* $y(x)$ at $x=1$ is approximately from 0.499978 to 0.500022, whereas the bounds given by the expressions above are 0.4 and 0.6.

7. *Another Type of Oscillating Series.* In the article mentioned above Hardy also discusses the series

$$M(a)=1/(1+a)-1/(1+pa)+1/(1+p^2a)-+\cdots, p>1.$$

He shows that this series likewise oscillates in the limit as a approaches zero along positive real values. The behavior of $M(a)$ near $a=0$ is closely analogous to that of $y(x)$ near $x=1$. In this connection it is interesting to note that the series $M(a)$ may be obtained from $y(x)$ by a rather simple transformation. The series

$$tx^{t-1}y(x)=t\{x^t-x^{t-1+p}+x^{t-1+p^2}-+\cdots\}, \qquad t>0 \text{ fixed},$$

is uniformly convergent in the closed interval $0\leqq x\leqq\xi<1$. Therefore we can write

$$t\int_0^\xi x^{t-1}y(x)\,dx = t\int_0^\xi x^t dx - t\int_0^\xi x^{t-1+p}dx + t\int_0^\xi x^{t-1+p^2}dx - +\cdots$$
$$= \frac{t\xi^{t+1}}{t+1} - \frac{t\xi^{t+p}}{t+p} + \frac{t\xi^{t+p^2}}{t+p^2} - +\cdots, \qquad 0\leqq\xi<1.$$

The terms of this last series are, however, even for $\xi=1$, not greater in absolute value than the corresponding terms of the convergent series of positive terms,

$$t/(t+1)+t/(t+p)+t/(t+p^2)+\cdots;$$

whence it appears that our series is uniformly convergent for $0 \leqq \xi \leqq 1$. We may therefore pass to the limit, and write

$$t \int_0^1 x^{t-1} y(x)\, dx = t/(t+1) - t/(t+p) + t/(t+p^2) - + \cdots.$$

If we now set $1/t = a$, we have the desired expression,

$$M(a) = 1/(1+a) - 1/(1+pa) + 1(1+p^2a) - + \cdots.$$

We proceed to develop some of the more interesting properties of this series. In the first place,

$$M(a/p^2) = 1/[1+(a/p^2)] - 1/[1+(a/p)] + M(a) > M(a),$$

and therefore the sequence, $M(a)$, $M(a/p^2)$, $M(a/p^4)$, $\cdots$, is monotone increasing. Since $M(a/p^{2n}) < 1$ for every n, it is also bounded above, and so possesses a limit. We set, $\lim_{n\to\infty} M(a/p^{2n}) = \mu(a)$. This limit-function plays a part similar to that of $F(x)$. It is evident that $\mu(a) = \mu(p^2a)$; and also, since

$$M(a/p^{2n}) = 1/[1+(a/p^{2n})] - M(a/p^{2n-1}) = 1/[1+(a/p^{2n})] - M(pa/p^{2n}),$$

we obtain, when n is allowed to increase indefinitely, the relation $\mu(a) = 1 - \mu(pa)$.

Again, since

$$\mu(a) = M(a) + \left[\frac{1}{1+a/p^2} - \frac{1}{1+a/p}\right] + \left[\frac{1}{1+a/p^4} - \frac{1}{1+a/p^3}\right] + \cdots,$$

and since

$$\frac{1}{1+a/p^{2n}} - \frac{1}{1+a/p^{2n-1}} = \left[1 - \frac{1}{1+a/p^{2n-1}}\right] - \left[1 - \frac{1}{1+a/p^{2n}}\right]$$

$$= \frac{a/p^{2n-1}}{1+a/p^{2n-1}} - \frac{a/p^{2n}}{1+a/p^{2n}} = \frac{1}{1+p^{2n-1}/a} - \frac{1}{1+p^{2n}/a},$$

we have

$$\mu(a) = M(a) + \left[\frac{1}{1+p/a} - \frac{1}{1+p^2/a}\right] + \left[\frac{1}{1+p^3/a} - \frac{1}{1+p^4/a}\right] + \cdots,$$

or $\mu(a) = M(a) + M(p/a)$. From this relation we deduce

$$\mu(a) = M(a) + M(p/a) = M(p/a) + M[p/(p/a)] = \mu(p/a).$$

By means of these equations we can readily find the value of $\mu(a)$ when

$a = 1$ or $a = p$. For, in general, $M(a) + M(pa) = 1/(1+a)$, and in particular $M(1) + M(p) = 1/(1+1) = 1/2$. It follows that

$$\mu(1) = \mu(p) = M(1) + M(p) = 1/2.$$

Thus, since $\mu(a) = 1 - \mu(pa)$, as $a \to 0$ along the sequence of values $1, 1/p, 1/p^2, \cdots, M(a)$ approaches the limit $1/2$. There is, of course, an analogous sequence of values of x along which $y(x) \to 1/2$.

Finally, if we let $a \to 0$ along the sequence $1/p^{1/2}, 1/p^{3/2}, 1/p^{5/2}, \cdots$, we may set $p = q^2$, and then $\mu(q) = M(q) + M(p/q) = 2M(q)$, which allows us to write a series for $\mu(q)$;

$$\mu(q) = 2\{1/(1+q) - 1/(1+q^3) + 1/(1+q^5) - + \cdots\}.$$

This series, however, although convergent for all $q > 1$, converges so slowly for values of q near unity that computation becomes practically impossible.

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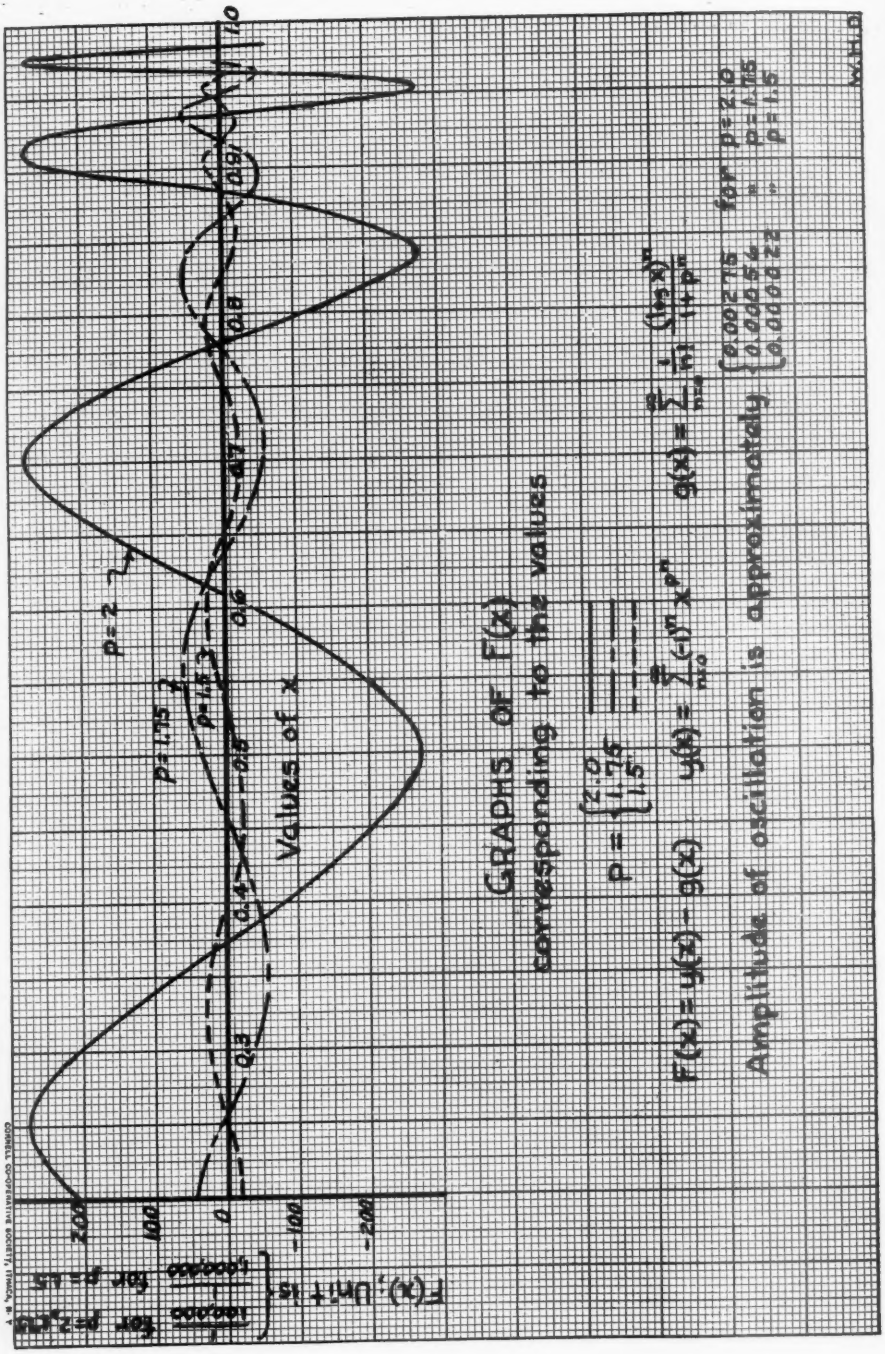
GRAPHS OF F(x)
corresponding to the values
p = 2.0, 1.75, 1.5
F(x) = y(x) − g(x)
y(x) = Σ (−1)^n x^p
g(x) = (2/π) Σ (1/n) sin(nx)/n^p
Amplitude of oscillation is approximately
0.09275 for p = 2.0
0.00654 " p = 1.75
0.00022 " p = 1.5
Values of x
F(x), Unit is 1/100,000 for p = 2.0
1/10,000 for p = 1.5
p = 2
p = 1.75
p = 1.5
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A NEW CALCULUS OF NUMERICAL FUNCTIONS.

By D. H. LEHMER.*

1. *Introduction.* In the theory of numerical functions there are two outstanding types of finite integration, namely

$$(1) \qquad \sum_{\nu=0}^{n} f(\nu) \qquad \text{and} \qquad (2) \qquad \sum_{\delta \mid n} f(\delta)$$

where δ runs over the divisors of n. The first of these processes leads to the calculus of finite differences, while the second one gives rise to a calculus of numerical functions which has been extensively studied.† If we make the sums (1) and (2) homogeneous as follows:

$$(3) \qquad \sum_{\nu=0}^{n} f(\nu)g(n-\nu) \qquad \text{and} \qquad (4) \qquad \sum_{\delta \mid n} f(\delta)g(n/\delta)$$

we obtain formulas for the n-th coefficient of the product of two power or Dirichlet series respectively. Both these sums are of the form $\sum f(a)g(b)$, where (a, b) are solutions of $\psi(a, b) = n$, for $\psi(a, b) = a + b$ and ab respectively.‡

It is the purpose of the present investigation to develop the fundamentals of a new calculus of numerical functions for which $\psi(a, b) = [a, b]$, the least common multiple of a and b. That is, we shall discuss sums of the form $\sum_n f(i)g(j)$, where it will be understood that for the subscript n, the sum extends over the solutions of $[i, j] = n$. Thus for $n = 6$

$$\begin{aligned}\sum_6 f(i)g(j) = {} & f(1)g(6) + f(2)g(3) + f(2)g(6) \\ & + f(3)g(2) + f(3)g(6) + f(6)g(1) \\ & + f(6)g(2) + f(6)g(3) + f(6)g(6).\end{aligned}$$

2. *The function $d(i, n)$.* In what follows we suppose that $n = \prod_{\nu=1}^{t} p_\nu^{a_\nu}$ where p_ν are distinct primes. The function $d(i, n)$ may be defined as follows: $d(i, n) = 0$, if i is not a divisor of n. Otherwise $d(i, n)$ is the largest divisor d of i for which n/d is prime to d. To obtain a formula for $d(i, n)$ suppose

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* National Research Fellow.

† Dickson, *History of the Theory of Numbers*, Vol. 1, Ch. 5, 10, 19.

‡ For another type of $\psi(x, y)$, see N. V. Bervy, *Matematicheskii Sbornik*, Vol. 18, pp 519-585.

that $i = \prod_{\nu=1}^{t} p_\nu^{\beta_\nu}$ where $0 \leqq \beta_\nu \leqq \alpha_\nu$, and that ν_k $(k = 1, 2, 3, \cdots)$ are those values of ν for which $\beta_\nu = \alpha_\nu$. Then $d(i, n) = \prod_{\nu=\nu_k} p_\nu^{\alpha_\nu}$. If $\alpha_\nu = 1$, $(\nu = 1, 2, \cdots t)$ then n is called a *simple number*. If n is simple, $d(i, n) = i$. If n is a power of a prime, then $d(i, n) = 1$ or n according as $i < n$, or $i = n$.

THEOREM 1. *If i_0 is a fixed divisor of n, then the solutions j of $[i_0, j] = n$ are $j = n/\delta$, where δ are the divisors of $d(i_0, n)$.*

Proof. If $i_0 = 1$, the theorem is obvious since $d(1, n) = 1$. For $i_0 > 1$, let us order the prime factors p_ν of $n = \prod p_\nu^{\alpha_\nu}$ in such a way that

$$i_0 = \prod_{\nu=1}^{h} p_\nu^{\alpha_\nu} \prod_{\nu=h+1}^{t} p_\nu^{\beta_\nu},$$

where $\beta_\nu < \alpha_\nu$. Any solution j of $[i_0, j] = n$, must be a divisor of n. Let $j = \prod_{\nu=1}^{t} p_\nu^{\gamma_\nu}$, then it follows from the definition of the L. C. M. that γ_ν must satisfy the following requirements

(I) $\quad \max.\ (\alpha_\nu, \gamma_\nu) = \alpha_\nu \qquad (\nu = 1, 2, \cdots, h)$

(II) $\quad \max.\ (\beta_\nu, \gamma_\nu) = \alpha_\nu \qquad (\nu = h+1, \cdots, t)$

(I) is satisfied for any γ_ν such that $0 \leqq \gamma_\nu \leqq \alpha_\nu$, while (II) requires $\gamma_\nu = \alpha_\nu$. Hence $j = \prod_{\nu=1}^{h} p_\nu^{\gamma_\nu} \prod_{\nu=h+1}^{t} p_\nu^{\alpha_\nu} = n/\delta$, where δ divides $\prod_{\nu=1}^{h} p_\nu^{\alpha_\nu} = d(i_0, n)$. Hence the theorem.

COROLLARY 1. *If $n = p^\alpha$, then*

$$(5) \qquad \Sigma_{p^\alpha} f(i) g(j) = f(p^\alpha) \sum_{s=0}^{\alpha-1} g(p^s) + g(p^\alpha) \sum_{s=0}^{\alpha-1} f(p^s) + f(p^\alpha) g(p^\alpha)$$

Proof. Since in this case $d(i, n) = 1$, or n according as $i < n$, or $i = n$, the corollary follows from the theorem by collecting the coefficients of $f(p^s)$.

3. *Factorable Functions.* A numerical function $f(n)$ is said to be factorable if $f(1) = 1$, and

$$f(n) f(m) = f(mn)$$

for every pair of coprime integers m, n. In particular

$$f(n) = \prod_{\nu=1}^{t} f(p_\nu^{\alpha_\nu}).$$

THEOREM 2. *If $f(n)$ and $g(n)$ are factorable functions then $F(n) = \Sigma_n f(i) g(j)$ is also factorable.*

Proof. Let m, n be any pair of integers for which $(m, n) = 1$, then

$$F(m)F(n) = \Sigma_m f(i_1)g(j_1)\ \Sigma_n f(i_2)g(j_2).$$

Since $(i_1, i_2) = (j_1, j_2) = 1$, it follows that

$$F(m)F(n) = \Sigma f(i_1 i_2)g(j_1 j_2). \tag{S}$$

Since also $(i_1, j_2) = (i_2, j_1) = 1$, we have

$$[i_1 i_2, j_1 j_2] = [i_1, j_1] \cdot [i_2, j_2] = mn.$$

Therefore every term of the above double sum (S) appears in

$$F(mn) = \Sigma_{mn} f(i)g(j). \tag{S'}$$

Conversely if $[i, j] = mn$, then $[(m, i), (m, j)] = m$ and $[(n, i), (n, j)] = n$. Hence if $i_1 = (m, i)$, $i_2 = (n, i)$, $j_1 = (m, j)$, $j_2 = (n, j)$, then $[i_1, j_1] = m$ and $[i_2, j_2] = n$, so that every term in (S') is in (S). Therefore

$$F(m)F(n) = \Sigma_{mn} f(i)g(j) = F(mn)$$

which is the theorem.

THEOREM 3. *If $f(n)$ and $g(n)$ are factorable functions, then*

$$\Sigma_n f(i)g(j) = \prod_{\nu=1}^{t} \{f(p_\nu{}^{a_\nu}) \sum_{s=0}^{a_\nu - 1} g(p_\nu{}^s) + g(p_\nu{}^{a_\nu}) \sum_{s=0}^{a_\nu} f(p_\nu{}^s)\}. \tag{6}$$

This follows immediately from Theorem 2, and Corollary 1.

4. *Integration.* The following lemma which properly belongs to section 8 is interpolated here to facilitate the proofs of Theorems 5, 9, 11, and 12.

LEMMA. *If $f(n)$, $g(n)$ and $h(n)$ are any numerical functions, then*

$$\begin{aligned} \Sigma_n f(i) \Sigma_j g(j_1)h(j_2) &= \Sigma_n g(i) \Sigma_j f(j_1)h(j_2) \\ &= \Sigma_n h(i) \Sigma f(j_1)g(j_2) \end{aligned}$$

where $[j_1, j_2] = j$.

Proof. Consider all one rowed matrices (i_1, i_2, i_3) for which $[i_1, i_2, i_3] = n$. Then it is seen that each of the sums of the theorem is equal to

$$\Sigma f(i_1)g(i_2)h(i_3) \qquad [i_1, i_2, i_3] = n.$$

Let us now consider in detail the case in which $g(n) = 1$, or in other words sums of the form

$$\Sigma_n f(i) = F(n). \tag{7}$$

The operation which carries f into F is a type of finite integration, analogous

to the well known types (1) and (2), and may be referred to as L. C. M. integration, or simply integration when no confusion can arise.

The sum of the type (2) may be reduced to the type (1) as follows

$$\sum_{\delta|n} f(\delta) = \sum_{\nu=1}^{n} f(\nu)\epsilon(n/\nu)$$

where $\epsilon(x)=1$, or 0 according as x is or is not an integer, but it is seldom advantageous to discuss a sum over divisors in this form. In a similar way a sum of the type (7) may be written

$$\Sigma_n f(i) = \sum_{\delta|n} f(\delta)\mu_1(d(\delta, n)) \tag{8}$$

where $\mu_1(n)$ is the number of divisors of n. Although this is a sum over divisors it is not always easily dealt with as such.

5. *The Inversion Function.* In the present theory the following function takes the place of Möbius' μ function.

THEOREM 4. *Let* $M(n)=\prod_{\nu=1}^{t}\{(\alpha_\nu+1)^{-1}-\alpha_\nu^{-1}\}$, $M(1)=1$, *then* $\Sigma_n M(i)=[1/n]$.

Proof. Obviously M is factorable and the theorem is true for $n=1$. If $n>1$, by (6)

$$\begin{aligned}\Sigma_n M(i) &= \prod_{\nu=1}^{t}\{M(1)+M(p_\nu)+\cdots+M(p_\nu^{\alpha_\nu-1})+(\alpha_\nu+1)M(p_\nu^{\alpha_\nu})\}\\ &= \prod_{\nu=1}^{t}\{1+\tfrac{1}{2}-1+\tfrac{1}{3}-\tfrac{1}{2}+\cdots\\ &\qquad +\alpha_\nu^{-1}-(\alpha_\nu-1)^{-1}+(\alpha_\nu+1)((\alpha_\nu+1)^{-1}-\alpha_\nu^{-1})\}=0;\end{aligned}$$

hence the theorem.

THEOREM 5. *If* $F(n)=\Sigma_n f(i)$, *then* $f(n)=\Sigma_n F(i)M(j)$.

Proof. $$\begin{aligned}\Sigma_n F(i)M(j) &= \Sigma_n M(j)\Sigma_i f(i_1)\\ &= \Sigma_n f(j)\Sigma_i M(i_1).\end{aligned}$$

This follows from the Lemma with i and j interchanged and one of the functions identically 1. But $\Sigma_i M(i_1)=[1/i]$. Hence the theorem.

This theorem is an analogue of Dedekind's inversion formula, and gives an explicit formula for the derivative of F.

6. *Successive Integrals.* Any numerical function determines a sequence

$$\cdots f_{-3}(n),\ f_{-2}(n),\ f_{-1}(n),\ f_0(n),\ f_1(n),\ f_2(n),\cdots \tag{9}$$

in which $f_{r+1}(n)=\sum_n f_r(i)$ or, what is the same, $f_{r-1}(n)=\sum_n f_n(i)M(j)$.

THEOREM 6. *If any f_r is factorable, so is every function of the sequence* (9).

This follows at once from Theorem 2.

THEOREM 7.* *Let f and g be two factorable functions and let*

$$f(p^\alpha)=\rho(p^{\alpha+1})-\rho(p^\alpha)$$
$$g(p^\alpha)=\theta(p^{\alpha+1})-\theta(p^\alpha),$$

$\rho(1)=\theta(1)=0$. *Finally if* $F(n)=\sum f(i)g(j)$ *then*

$$F(p^\alpha)=\rho(p^{\alpha+1})\theta(p^{\alpha+1})-\rho(p^\alpha)\theta(p^\alpha).$$

Proof. By Theorem 3,

$$\begin{aligned}F(p^\alpha)&=\{\rho(p^{\alpha+1})-\rho(p^\alpha)\}\{1+\sum_{s=1}^{\alpha-1}(\theta(p^{s+1})-\theta(p^s))\}\\&\quad+\{\theta(p^{\alpha+1})-\theta(p^\alpha)\}\{1+\sum_{s=1}^{\alpha}(\rho(p^{s+1})-\rho(p^s))\}\\&=\{\rho(p^{\alpha+1})-\rho(p^\alpha)\}\,\theta(p^\alpha)+\{\theta(p^{\alpha+1})-\theta(p^\alpha)\}\,\rho(p^{\alpha+1})\\&=\rho(p^{\alpha+1})\theta(p^{\alpha+1})-\rho(p^\alpha)\theta\,(p^\alpha),\end{aligned}$$

which is the theorem.

To study the properties of the general sequences (9) it is first necessary to consider the special case in which $f_0(n)=[1/n]$. The r-th member of this special sequence we denote by $M_r(n)$.

THEOREM 8. *If* $M_0(n)=[1/n]$, *then* $\prod_{\nu=1}^{t}\{(\alpha_\nu+1)^r-\alpha_\nu{}^r\}=M_r(n)$.

Proof. In Theorem 7 put $\theta(p^\alpha)=\alpha$. Then $g(n)=1$. Assume the above expression for $M_r(n)$. Then if we set $\rho(p^\alpha)=\alpha^r$ we have $f(n)=M_r(n)$ and by Theorem 7, $M_{r+1}(p^\alpha)=\sum_{p^\alpha}M_r(i)=(\alpha+1)^{r+1}-\alpha^{r+1}$. If we put $\theta(p^\alpha)=\alpha^{-1}$, then $g(n)=M(n)$ and for the same choice of $\rho(p^\alpha)$, $M_{r-1}(p^\alpha)=\sum_{p^\alpha}M_r(i)M(j)=(\alpha+1)^{r-1}-\alpha^{r-1}$. But the theorem is true for $r=0$. Hence the induction is complete. It is to be observed that

$$M(n)=M_{-1}(n),\qquad M_r(1)=1.$$

The formula for $M_r(n)$ must not be used for $n=1$, $r<1$.

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* An equivalent statement of this theorem: $\Sigma F(\delta)=\Sigma f(\delta)\Sigma g(\delta)$, $\delta \mid n$, has been given by von Sterneck, *Monatshefte für Mathematik und Physik*, Vol. 5 (1894), pp. 255-266. In this form the theorem is true for non-factorable functions as well. Von Sterneck's proof is not adequate however. A rigorous proof of his general theorem will appear in a forthcoming paper.

THEOREM 9. *If $f_0(n)$ is any numerical function, then*

$$f_r(n) = \sum_n f_0(i) M_r(j).$$

Proof. The theorem is trivial for $r = 0$. Assuming it true for $f_r(n)$, we may write, (using the lemma with $g(n) = 1$)

$$\begin{aligned} f_{r+1}(n) &= \sum_n f_r(j) = \sum_n \sum_j f_0(j_1) M_r(j_2) \\ &= \sum_n f_0(i) \sum_j M_r(j_1) = \sum_n f_0(i) M_{r+1}(j). \end{aligned}$$

For $f_{r-1}(n)$ we have

$$\begin{aligned} f_{r-1}(n) &= \sum_n M_{-1}(i) f_r(j) = \sum_n M_{-1}(i) \sum_j f_0(j_1) M_r(j_2) \\ &= \sum_n f_0(i) \sum_j M_{-1}(j_1) M_r(j_2) = \sum_n f_0(i) M_{r-1}(j). \end{aligned}$$

This proves the theorem.

Since $f_r(n)$ defines the same sequence (9) as does $f_0(n)$ we may put the preceding theorem into the following form.

THEOREM 10. *If $r = r_1 + s_1 = r_2 + s_2$ be any two partitions of r, then*

$$f_r(n) = \sum_n f_{r_1}(i) M_{s_1}(j) = \sum_n f_{r_2}(i) M_{s_2}(j).$$

More generally we have

THEOREM 11. *If $r = r_1 + s_1 = r_2 + s_2$, then for any pair of numerical functions $f_0(n)$ and $g_0(n)$,*

$$\sum_n f_{r_1}(i) g_{s_1}(j) = \sum_n f_{r_2}(i) g_{s_2}(j).$$

Proof.

$$\begin{aligned} \sum_n f_{r_1}(i) g_{s_1}(j) &= \sum_n f_{r_1}(i) \sum_j g_0(j_1) M_{s_1}(j_2) \\ &= \sum_n g_0(i) \sum_j f_{r_1}(j_1) M_{s_1}(j_2) = \sum_n g_0(i) f_r(j). \end{aligned}$$

The last two equalities follow from the Lemma and from Theorem 9. Since the last result does not depend on r_1 and s_1, but only on their sum, the theorem follows.

The preceding theorem justifies the following notation

$$F_{r+s}(n) = \sum_n f_r(i) g_s(j).$$

As $r + s$ varies we obtain a sequence

$$(10) \quad \ldots F_{-3}(n), F_{-2}(n), F_{-1}(n), F_0(n), F_1(n), F_2(n), F_3(n), \cdots.$$

The relation between any two elements of the sequence (10) is the same as that between corresponding elements of (9). To show this it is sufficient to prove the following:

THEOREM 12. *If* $F_{r+s}(n)=\sum_n f_r(i)g_s(j)$ *where* f_0 *and* g_0 *are two arbitrary functions, then*

$$F_{r+1}(n)=\sum_n F_r(i).$$

Proof. By definition

$$\begin{aligned} F_{r+1}(n)&=\sum_n f_0(i)g_{r+1}(j)=\sum_n f_0(i)\sum_j g_r(j_1)\\ &=\sum_n\sum_j f_0(j_1)g_r(j_2)=\sum_n F_r(j), \end{aligned}$$

which is the theorem.

7. ***Multiple Integrals.*** It is possible to give another interpretation to the sequence (9) for $r>0$ by considering all the one rowed matrices $(i_0, i_1, i_2, \cdots, i_r)$ of positive integers whose L. C. M. $[i_0, i_1, i_2, \cdots, i_r]=n$.

THEOREM 13. *If* $f_r(n)$, $r>0$, *is a member of the sequence* (9), *then*

$$f_r(n)=\sum f_0(i_0), \qquad [i_0, i_1, \cdots, i_r]=n.$$

Proof. The theorem is obvious for $r=1$ as it reduces to the definition of $f_1(n)$. Assuming the theorem true for $f_r(n)$ we have

$$\begin{aligned} f_{r+1}(n)&=\sum_n f_r(i)\\ &=\sum_n\sum f_0(i_0) \qquad [i_0, i_1, \cdots, i_r]=i \qquad [i, j]=n. \end{aligned}$$

If we write $j=i_{r+1}$, we have

$$n=[i, i_{r+1}]=[[i_0, i_1, \cdots, i_r], i_{r+1}]=[i_0, i_1, \cdots, i_{r+1}].$$

Then $$f_{r+1}(n)=\sum f_0(i_0) \qquad [i_0, i_1, \cdots, i_{r+1}]=n.$$

This completes the induction.

If we set $f_0(n)=1=M_1(n)$, we find by applying Theorem 13 that the number of solutions of

$$[i_0, i_1, i_2, \cdots, i_r]=n$$

is $M_{r+1}(n)$. These matrices correspond to the r-th divisors * of n in the calculus of divisors.

8. *General r-th order Inversion.* From Theorems 9 and 13 we obtain the following generalization of Theorem 5.

THEOREM 14. *If*

$$F(n)=\sum f(i_0) \qquad [i_0, i_1, \cdots, i_r]=n$$

then

$$f(n)=\sum F(i_0)M(i_1)M(i_2)\cdots M(i_r). \tag{12}$$

* *American Journal of Mathematics*, Vol. 52 (1930), pp. 293-304.

Proof. If we write f_0 for f, then $F = f_r$ by Theorem 13.

Since $[i_0, i_1, \cdots, i_{r-1}, i_r] = [i_0, [i_1, [i_2, [\cdots, [i_{r-1}, i_r]]\cdots]$, we may replace $[i_{k+1}, i_{k+2}, \cdots, i_r]$ by j_k, $k = 0, 1, \cdots, r-2$, and have $[i_k, j_k] = j_{k-1}$. In particular $[i_0, j_0] = n$. The sum in (12) may be written

$$\Sigma_n f_r(i_0)\Sigma_{j_0} M(i_1)M(i_2)\cdots M(i_r) =$$

$$(13) \qquad \Sigma_n f_r(i_0)\Sigma_{j_0} M(i_1)\Sigma_{j_1} M(i_2)\cdots\Sigma_{j_{r-3}} M(i_{r-2})\Sigma_{j_{r-2}} M(i_{r-1})M(i_r).$$

But the very last sum has the value $M_{-2}(j_{r-2})$. The last two sums therefore may be replaced by

$$\Sigma_{j_{r-3}} M(i_{r-2})M_{-2}(j_{r-2}) = M_{-3}(j_{r-3}).$$

Continuing this process the whole sum (13) is seen to collapse to

$$\Sigma_n f_r(i_0)M_{-r}(j_0) = f_0(n)$$

by Theorem 9. This proves Theorem 14.

9. *L. C. M. Multiplication.* The sums (11) and (12) are the simplest examples of the general type

$$(14) \qquad F(n) = \Sigma f_1(i_1)f_2(i_2)f_3(i_3)\cdots f_r(i_r)$$

where $f_v(n)$ are any numerical functions (not necessarily belonging to a sequence (9)). The function F might be called the "L. C. M. product" * of the f's, and we might write $F = f_1 \cdot f_2 \cdot f_3 \cdots f_r$. Such a symbolic product enjoys all the properties of ordinary multiplication and with this operation as basis, it is possible to construct a theory of numerical functions of which the present paper is but a small part.

10. *Examples.* The following examples illustrate the nature of the functions in the sequence (9), which arise from a number of familiar factorable functions.

	$f_0(n)$	$f_r(p^\alpha)$
I	n^k	$p^{k\alpha}\alpha^r + M_r(p^\alpha)\sigma_k(p^\alpha)$
II	$\lambda(n)$	$\begin{cases} -\alpha^r & \alpha \text{ odd} \\ (\alpha+1)^r & \alpha \text{ even} \end{cases}$
III	$\mu(n)$	$\mu(p^\alpha)$
IV	$\phi^{(k)}(n)$	$p^{\alpha k}(\alpha+1)^r - p^{(\alpha-1)k}\alpha^r$
V	$2^{t(n)}$	$2\alpha^r + M_r(p^\alpha)M_2(p^\alpha)$
VI	$2^{t(n)}\lambda(n)$	$\lambda(p^\alpha)M_{2r}(p^\alpha)/M_r(p^\alpha)$.

* This is the analogue of what E. T. Bell designates ideal product in the divisor theory. Compare: *University of Washington Publications*, Vol. 1, No. 1, pp. 9, 10.

To illustrate L. C. M. products we give the following examples:

VII $\quad \sum_n 2^{t(i)}\mu_1(j) = \mu_1(n)\mu_1(n^3)$

VIII $\quad \sum_n 2^{t(i)}\lambda(i)\lambda(j) = \lambda(n)$

IX $\quad \sum_n 2^{t(i)}\lambda(i)2^{t(j)} = \lambda(n)\prod_{\nu=1}^{t(n)} 4\alpha_\nu$

X $\quad \sum_n f(i)\mu^2(j) = \prod_{\nu=1}^{t}\{2f(p_\nu{}^{a_\nu}) + \mu^2(p_\nu{}^{a_\nu})\},$

where f is any factorable function. The next 5 examples illustrate the L. C. M. powers of certain functions.

XI $\quad \sum_n \mu_1(i)\mu_1(j) = \mu_1{}^3(n)$

XII $\quad \sum_n 2^{t(i)}2^{t(j)} = 2^{t(n)}\mu_1(n^2)$

XIII $\quad \sum_n 2^{t(i)}\lambda(i)2^{t(j)}\lambda(j) = [1/n]$

XIV $\quad \sum\{\mu(i_0)\mu(i_1)\cdots\mu(i_r)\}^2 = \mu^2(n)M_{r+1}(n)$

XV $\quad \sum\phi(i_0)\varphi(i_1)\cdots\phi(i_r) = \phi^{(r+1)}(n).$

In the last two formulas $[i_0, i_1, \cdots, i_r] = n$, $(r > 0)$. In the above table $\lambda(n) = \prod_{\nu=1}^{t(n)}(-1)^{a_\nu}$ is Liouville's function, $\sigma_k(n)$ is the sum of k-th powers of the divisors of n, and $\phi^{(k)}(n) = n^k\prod_{\nu=1}^{t}(1 - p_\nu{}^{-k})$ is Jordan's generalized totient function.

11. *Periodic Sequences.* Thus far we have developed features of our calculus which have precise analogues in the divisor calculus. It would be misleading not to consider some of the essential dissimilarities of the two theories. The first of these has to do with periodicities in the sequence (9), and its divisor theory analogue,* namely the sequence obtained from $f_0(n)$ by repeated application of the operation (2). One may show that no sequence of this latter type is periodic except for the trivial case of $f_0(n) \equiv 0$. For the sequence (9) however we see from example III that the sequence defined by $f_0(n) = \breve{\mu}(n)$ is periodic with the period 1. To show that this is essentially the only periodic sequence, we use

THEOREM 15. *If $f(n)$ satisfies the following conditions.*

(15) $\quad f(n) = \sum_n f(i)M_r(j) \qquad (r \neq 0)$

(16) $\quad f(1) = 0$

then $f(n)$ vanishes identically.

---

* For a detailed account of this sequence see *American Journal of Mathematics*, Vol. 52 (1930), pp. 296-299.

Proof. Suppose that $f(n) \not\equiv 0$. Then an integer L exists such that

$$f(\nu) = 0, \quad \nu < L, \quad f(L) \not\equiv 0, \quad L > 1. \tag{17}$$

In (15) set $n = L$, then in view of (17)

$$f(L) = \Sigma_L f(i) M_r(j) = f(L) \sum_{\delta | L} M_r(\delta). \tag{18}$$

But

$$\sum_{\delta | L} M_r(\delta) = \prod_{\nu=1}^{t} \sum_{s=0}^{\alpha_\nu} M_r(p^s) = \prod_{\nu=1}^{t} (\alpha_\nu + 1)^r = \mu_1{}^r(L).$$

Since $r \not\equiv 0$, (18) implies that $\mu_1(L) = 1$, which contradicts the hypothesis $L > 1$. Hence the theorem.

THEOREM 16. *If* $f(n) = \Sigma_n f(i) M_r(j)$ $r \not\equiv 0$ *then* $f(n) = f(1)\mu(n)$.

Proof. Let $f(n) - f(1)\mu(n) = \chi(n)$ *then*

$$\begin{aligned} \Sigma_n \chi(i) M_r(j) &= \Sigma_n f(i) M_r(j) - f(1) \Sigma_n \mu(i) M_r(j) \\ &= f(n) - f(1)\mu(n) = \chi(n) \end{aligned}$$

by hypothesis and example III. Moreover $\chi(1) = 0$ and therefore by Theorem 15, $\chi(n)$ vanishes identically. Hence the theorem follows.

Thus if any two members of a sequence (9) are equal, all members are equal and are proportional to the μ function. This property of μ is only a special case of

THEOREM 17. *Let* $g(n)$ *be any numerical function for which* $g(1) = 1$, *then* $\Sigma_n g(i)\mu(j) = \mu(n)$.

Proof. By Theorem 1

$$\Sigma_n g(i)\mu(j) = \sum_{i | n} g(i) \sum_{\delta | d(i,n)} \mu(n/\delta).$$

Since μ is factorable $\mu(n/\delta) = \mu(n/d(i,n))\mu(d(i,n)/\delta)$. Hence

$$\begin{aligned} \Sigma_n g(i)\mu(j) &= \sum_{i | n} g(i)\mu(n/d(i,n)) \sum_{\delta | d(i,n)} \mu(\delta) \\ &= \sum_{i | n} g(i)\mu(n/d(i,n))[1/d(i,n)]. \end{aligned}$$

It follows that the coefficient of $g(i)$ is 0 or $\mu(n)$ according as $d(i,n) > 1$, or $= 1$. Hence, if n is not simple, $\mu(n) = 0$ and the whole sum vanishes proving the theorem. If n is simple we have seen that $d(i,n) = i$. In this case the only non-zero term is that for which $i = 1$. Hence $\Sigma_n g(i)\mu(j) = g(1)\mu(n) = \mu(n)$. This proves the theorem. By Theorem 16, μ is the only function which satisfies Theorem 17.

COROLLARY. *If $g(n)$ is any function at all, then*

$$\Sigma_n g(i)\mu(j) = g(1)\mu(n). \tag{19}$$

12. *The Equation $f \cdot g = h$.* In the preceding section we considered two special cases of the equation

$$\Sigma_n f(i)g(j) = h(n) \tag{20}$$

in which $h(n) = f(n)$. The equation (20) has other special cases which should be compared with those of its analogue

$$\sum_{\delta|n} f(\delta)g(n/\delta) = h(n). \tag{21}$$

For brevity we shall write (20) in the form $f \cdot g = h$, adopting the notation of section 8. Likewise (21) will be written $fg = h$.

Case I. $f = g = h$. It may be shown that the only solutions of $ff = f$ are $f(n) \equiv 0$, $f(n) = [1/n]$. The equation $f \cdot f = f$ has however an infinity of solutions. To exhibit infinitely many factorable solutions it is sufficient to select any finite or infinite increasing sequence $\{\nu_k\}$ $(k = 1, 2, 3, \cdots)$ of non-negative integers. Define $f(n)$ as that factorable function for which

$$f(p^\alpha) = \begin{cases} (-1)^k, & \alpha = \nu_k \\ 0, & \text{otherwise.} \end{cases}$$

Then by (5)

$$\Sigma_{p^\alpha} f(i)f(j) = f^2(p^\alpha) + 2f(p^\alpha) \sum_{s=0}^{\alpha-1} f(p^s). \tag{22}$$

If α is not a member of the sequence $\{\nu_k\}$, the right hand member of (22) vanishes. Otherwise it becomes for $\alpha = \nu_k$

$$1 + 2(-1)^k \sum_{a=0}^{k-1} (-1)^a = 1 + 2(-1)^k (1 - (-1)^k)/2 = (-1)^k = f(p^{\nu_k})$$

Therefore $f \cdot f = f$. Special examples of these functions are $[1/n]$, $\mu(n)$, $\lambda(n)$. There are also infinitely many non-factorable functions for which $f \cdot f = f$.

Case II. $f = g$, $h(n) = [1/n] = M_0(n)$. The equation $ff = M_0$ has only two solutions $f(n) = \pm M_0(n)$. But $f \cdot f = M_0$ has infinitely many other solutions. To exhibit infinitely many factorable solutions choose any function $\theta(n)$ for which $\theta(1) = 1$, $\theta(n) = \pm 1$ $(n > 1)$. Define $f(n)$ as that factorable function for which

$$f(p^\alpha) = \theta(\alpha + 1) - \theta(\alpha), \qquad \alpha > 0.$$

Then by Theorem 7

$$\sum_{p^\alpha} f(i)f(j) = \theta^2(\alpha+1) - \theta^2(\alpha) = 0.$$

Obviously for $n=1$, $f \cdot f = 1$. Hence for any such f, $f \cdot f = M_0$. The only never vanishing factorable solution of this equation is $f(n) = 2^{t(n)}\lambda(n)$, obtained from $\theta(n) = (-1)^{n+1}$ (Example XIII).

Case III. $h=0$. The equation $fg=0$ implies either $f(n)=0$ or $g(n)=0$, or both. However one may find infinitely many solutions of $f \cdot g = 0$ in which neither function is identically zero. In fact it follows from the corollary of Theorem 17 that if $f=\mu$, then $f \cdot g = 0$ is satisfied by every function g, which vanishes for $n=1$.

The discussion of the infinite series aspect of the above theory will be included in a paper on infinite series in general.

ON CERTAIN IDENTITIES IN THETA FUNCTIONS.

By Russell Smith Park.

1. *Introduction.* The object of this paper is to obtain certain identities in Jacobi's theta functions, by a method essentially the same as that used by Latimer in his paper, "On Certain Identities in Theta Functions." * In that paper sets of integral elements in generalized quaternion algebras were used in which all the parameters α, β, γ were odd. It is proposed in this paper to obtain similar identities when $\gamma = 2\sigma$, where σ is an odd integer. The sets of integral elements, to be used in obtaining the first identities, were obtained by Miss Darkow † for a generalized quaternion algebra A. The elements to be used in obtaining the last two identities are algebraic numbers.

2. *Change of notation and definition of sets.* Consider the rational algebra A_1 with the basal numbers

$$1, \qquad I = (\beta\gamma)^{\frac{1}{2}}i, \qquad J = (\gamma\alpha)^{\frac{1}{2}}j, \qquad K = (\alpha\beta)^{\frac{1}{2}}k,$$

where $\gamma = 2\sigma$ and α, β, σ are positive odd integers such that $\alpha\beta\sigma$ contains no square factor > 1, and i, j, k are the usual quaternion units.

In Miss Darkow's algebra A the multiplication table involved two even parameters α and β which were factored into

$$\alpha = 2\mu\delta, \qquad \beta = 2\nu\delta,$$

where μ, ν, δ were odd integers and 2δ was the positive g. c. d. of α and β. If we assume that $\alpha < 0$, $\beta > 0$, and in the multiplication table of A replace μ, ν, δ, e_1, e_2, e_3 by $-\beta$, α, σ, I, J, $2\sigma K$ respectively it will be seen that A is equivalent to A_1. The multiplication table of A_1 is

$$I^2 = -\beta\gamma, \qquad J^2 = -\gamma\alpha, \qquad K^2 = -\alpha\beta,$$
$$IJ = -JI = \gamma K, \quad JK = -KJ = \alpha I, \quad KI = -IK = \beta J.$$

Let $\alpha = A\alpha'$, $\beta = B\beta'$, $\sigma = C\sigma'$ where A, B, C are defined as follows. A is the product of all the prime factors A_i of α such that $-2\beta\sigma$ is a quadratic non-residue of every A_i; or $A = 1$ if α contains no such factor. Let B, C be similarly defined by a cyclic interchange of A, B, C; A_i, B_i, C_i; α, β, 2σ.

* *Transactions of the American Mathematical Society*, Vol. 42 (1930), pp. 832-846.

† *Annals of Mathematics*, Vol. 28 (1926-27), pp. 263-270.

The properties of the integral elements of A obtained by Miss Darkow will be enumerated in the new notation. There are odd integers E_i, F_i, H_i prime to β', α', σ' such that

$$1 + 2\alpha\sigma E_i^2 \equiv 0 \bmod \beta', \quad 1 + 2\beta\sigma F_i^2 \equiv 0 \bmod \alpha', \quad 1 + \alpha\beta H_i^2 \equiv 0 \bmod \sigma'.$$

Define the class ijk of algebras A_1 as the class of algebras for which

$$-\beta\sigma \equiv i \bmod 8, \quad \alpha\sigma \equiv j \bmod 8, \quad -\alpha\beta \equiv k \bmod 8.$$

Since $ij \equiv k \bmod 8$, algebras of four types are defined.

A_1 of type	A_1 of class.			
A	155	375	515	735
B	133	357	573	717
C	111	331	551	771
D	177	313	537	753

Define set S' as the totality of elements in the form

$$\Omega = \tfrac{1}{2}\left[\xi - \frac{\eta}{2\beta'\sigma'} I + \frac{\zeta}{2\alpha'\sigma'} J - \frac{\lambda}{\alpha'\beta'} K\right] \tag{1}$$

where ξ, η, ζ, λ are integers which satisfy one congruence of each pair of the equivalent congruences

$$\begin{aligned} &\eta \equiv 2A\sigma' E\lambda \quad \text{or} \quad \lambda \equiv -\alpha' CE\eta \bmod \beta', \\ &\zeta \equiv 2B\sigma' F\lambda \quad \text{or} \quad \lambda \equiv -\beta' CF\zeta \bmod \alpha', \\ &\zeta \equiv B\alpha' H\eta \quad \text{or} \quad \eta \equiv -\beta' AH\zeta \bmod \sigma', \end{aligned} \tag{2}$$

and also satisfy condition (3) below. E, F, H are arbitrarily chosen E_i, F_i, H_i mentioned above. Let S'', S''', S^{IV}, S^{V}, S^{VI} be defined in the same way except that instead of (3) we employ (4), (5), (6), (7), (8) respectively.

$$\xi \equiv \lambda \bmod 2, \quad \eta, \zeta \text{ even}, \quad \eta \equiv \zeta \bmod 4; \tag{3}$$

$$\begin{aligned} &\xi \equiv \lambda \equiv 1 \bmod 2, \quad \eta, \zeta \text{ even}, \quad \zeta \equiv \eta + 2 \bmod 4, \\ &\text{or} \quad \xi \equiv \lambda \equiv 0 \bmod 2, \quad \eta, \zeta \text{ even}, \quad \eta \equiv \zeta \bmod 4; \end{aligned} \tag{4}$$

$$\xi \equiv \lambda \bmod 2, \qquad \eta \equiv \zeta \bmod 4; \tag{5}$$

$$\begin{aligned} &\xi \equiv \lambda \equiv 0 \bmod 2, \quad \eta, \zeta \text{ even}, \quad \eta \equiv \zeta \bmod 4, \\ &\text{or} \quad \xi \equiv \lambda \equiv 1 \bmod 2, \quad \eta, \zeta \text{ even}, \quad \eta \equiv \zeta + 2 \bmod 4, \\ &\text{or} \quad \xi \text{ even}, \qquad \eta, \zeta, \lambda \text{ odd}, \qquad \eta \equiv \zeta \bmod 4, \\ &\text{or} \quad \lambda \text{ even}, \qquad \xi, \eta, \zeta \text{ odd}, \quad \eta \equiv \zeta + 2 \bmod 4; \end{aligned} \tag{6}$$

$$(7)\qquad \xi \equiv \lambda \bmod 2,\quad \eta, \zeta \text{ even},\quad \eta \equiv \zeta \bmod 4,$$
$$\text{or}\qquad \xi \equiv \lambda \bmod 2,\quad \eta, \zeta \text{ odd},\quad \eta \equiv \zeta + 2 \bmod 4;$$

$$(8)\qquad \xi \equiv \lambda \equiv 0 \bmod 2,\quad \eta, \zeta \text{ even},\quad \eta \equiv \zeta \bmod 4,$$
$$\text{or}\quad \xi \equiv \lambda \equiv 1 \bmod 2,\quad \eta, \zeta \text{ even},\quad \eta \equiv \zeta + 2 \bmod 4,$$
$$\text{or}\quad \xi \text{ even},\quad \eta, \zeta, \lambda \text{ odd},\quad \eta \equiv \zeta + 2 \bmod 4,$$
$$\text{or}\quad \lambda \text{ even},\quad \xi, \eta, \zeta \text{ odd},\quad \eta \equiv \zeta \bmod 4.$$

Let E', E'', E''', E^{IV} be integers, congruent mod β' to E, such that

$$A\sigma'E' \equiv \alpha' CE'' \equiv 3A\sigma'E''' \equiv 3\alpha' CE^{IV} \equiv 1 \bmod 8.$$

Let F', F'', $\cdots$, H^{IV} satisfy the conditions

$$B\sigma'F' \equiv B\alpha'H' \equiv 7\beta' CF'' \equiv 7\beta' AH''$$
$$\equiv 3B\sigma'F''' \equiv 3B\alpha'H''' \equiv 5\beta' CF^{IV} \equiv 5\beta' AH^{IV} \equiv 1 \bmod 8,$$

where each $F^{(i)}$ is congruent mod α' to F and each $H^{(i)}$ is congruent mod σ' to H.

3. *Construction of the theta functions corresponding to sets $S^{(i)}$ and derivation of the identities.* It may be shown that η, ζ, λ satisfy (2) and (3) if and only if they may be written in form (9) below with r and p subject to the indicated inequalities. Similarly η, ζ, λ satisfy (2) and (4), (5), or (6) if and only if they may be written in the forms (10), (11), or (12) respectively. Let (11′), (12′) be the forms obtained from (11), (12) respectively by replacing E', F', H', E'', F'', H'', by E''', F''', H''', E^{IV}, F^{IV}, H^{IV} respectively. Then it may be shown that η, ζ, λ satisfy (2) and (7) or (8) if and only if they may be written in the forms (11′) or (12′) respectively.

$$(9)\qquad \begin{aligned} \lambda &= 2\alpha'\beta' m_1 + r, \\ \eta &= 4\beta'\sigma' m_2 + 2A\sigma' Er - 2\alpha\beta' Hp, \\ \zeta &= 4\alpha'\sigma' m_3 + 2B\sigma' Fr + 2\alpha' p, \\ & 0 \leqq r < 2\alpha'\beta', \quad 0 \leqq p < 2\sigma'. \end{aligned}$$

$$(10)\qquad \begin{aligned} \lambda &= 2\alpha'\beta' m_1 + r, \\ \eta &= 4\beta'\sigma' m_2 + 2(\beta' + AE)\sigma' r - 2\alpha\beta' Hp, \\ \zeta &= 4\alpha'\sigma' m_3 + 2B\sigma' Fr + 2\alpha' p, \\ & 0 \leqq r < 2\alpha'\beta', \quad 0 \leqq p < 2\sigma'. \end{aligned}$$

$$(11)\qquad \begin{aligned} \lambda &= 2\alpha'\beta' m_1 + r, \\ \eta &= 4\beta'\sigma' m_2 + 2A\sigma' E'r - \alpha\beta' H'p, \\ \zeta &= 4\alpha'\sigma' m_3 + 2B\sigma' F'r + \alpha' p, \\ & 0 \leqq r < 2\alpha'\beta', \quad 0 \leqq p < 4\sigma'. \end{aligned}$$

$$
\begin{aligned}
&\lambda = 2\alpha'\beta' m_1 + r,\\
(12)\qquad &\eta = 4\beta'\sigma' m_2 + 2(\beta' + AE'')\sigma' r - (\alpha H'' - 2\sigma')\beta' p,\\
&\zeta = 4\alpha'\sigma' m_3 + 2B\sigma' F'' r + \alpha' p\\
&\qquad 0 \leqq r < 2\alpha'\beta', \qquad 0 \leqq p < 4\sigma'.
\end{aligned}
$$

Let q be a complex number, $|q| < 1$; $a = A/\alpha'$, $b = B/\beta'$, $c = C/2\sigma'$ and

$$
\begin{aligned}
F(w;r) &= \sum_{m_1} q^{ab\lambda^2/4} e^{ab\lambda w i},\\
G_j(y;r,p) &= \sum_{m_2} q^{bc\eta^2/4} e^{bc\eta y i},\\
H_j(z;r,p) &= \sum_{m_3} q^{ac\zeta^2/4} e^{ac\zeta z i}, \qquad (j = 1, 2 \cdots 6),
\end{aligned}
$$

where the summation extends from $-\infty$ to $+\infty$; r, p are integers; $\lambda = 2\alpha'\beta' m_1 + r$; η, ζ are given by (9), (10), (11), (12), (11′) or (12′) according as $j = 1$, 2, 3, 4, 5, or 6. We may write

$$
F(w;r) = q^{abr^2/4} e^{abrwi}\vartheta_3[AB(w - rki), q^{ab}],
$$

where $2k = \log q$, and ϑ_3 is Jacobi's theta function. Hence F is absolutely convergent. The other twelve functions may be similarly expressed in terms of theta functions and hence are absolutely convergent. Let s, p be integers such that $0 \leqq s < \alpha'\beta'$, $0 \leqq p < 2\sigma'$. Then

$$
\begin{aligned}
\psi_{11}(s,p) &\equiv \vartheta_3(x) F(w;2s)\, G_1(y;2s,p)\, H_1(z;2s,p)\\
&= \sum q^{(\xi^2 + bc\eta^2 + ac\zeta^2 + ab\lambda^2)/4} e^{(\xi x + bc\eta y + ac\zeta z + ab\lambda w)i}
\end{aligned}
$$

where the summation extends over all quadruples ξ, η, ζ, λ such that

$$
\xi \equiv \lambda \equiv 0 \bmod 2, \quad \eta, \zeta \text{ even}, \quad \eta \equiv \zeta \bmod 4,
$$

and such that η, ζ, λ may be written in form (9) with r replaced by $2s$. For every term of ψ_{11} there is a corresponding element of S'. Conversely, if Ω is an element of S', with ξ even, there is a uniquely determined pair of integers s, p; $0 \leqq s < \alpha'\beta'$; $0 \leqq p < 2\sigma'$, such that ψ_{11} contains exactly one term corresponding to Ω. Let

$$
\psi_{12}(s,p) \equiv \vartheta_2(x) F(w;2s+1)\, G_1(y;2s+1,p)\, H_1(z;2s+1,p)
$$

and repeat the above argument except that

$$
\xi \equiv \lambda \equiv 1 \bmod 2, \quad \eta, \zeta \text{ even}, \quad \eta \equiv \zeta \bmod 4.
$$

Then

$$
\Phi_1(x,y,z,w) \equiv \sum_{s=0}^{\alpha'\beta'-1} \sum_{p=0}^{2\sigma'-1} [\psi_{11} + \psi_{12}] = \sum_{\Omega} q^{N(\Omega)} e^{2R(\Omega X')i}
$$

where, assuming for the present that x, y, z, w are real

$$X = x - \frac{y}{2\beta'\sigma'} I + \frac{z}{2\alpha'\sigma'} J - \frac{w}{\alpha'\beta'} K;$$

$R(\Omega X')$ is the real part of the quaternion $\Omega X'$; and the summation extends without omission or duplication over the elements of S'.

Let

$$\psi_{21}(s,p) \equiv \vartheta_2(x) F(w; 2s+1) G_2(y; 2s+1, p) H_2(z; 2s+1, p),$$
$$\psi_{22}(s,p) \equiv \vartheta_3(x) F(w; 2s) G_2(y; 2s, p) H_2(z; 2s, p),$$
$$\psi_{31}(s,p) \equiv \vartheta_2(x) F(w; 2s+1) G_3(y; 2s+1, p) H_3(z; 2s+1, p),$$
$$\psi_{32}(s,p) \equiv \vartheta_3(x) F(w; 2s) G_3(y; 2s, p) H_3(z; 2s, p),$$
$$\psi_{41}(s,t) \equiv \vartheta_3(x) F(w: 2s) G_4(y; 2s, 2t) H_4(z; 2s, 2t),$$
$$\psi_{42}(s,t) \equiv \vartheta_2(x) F(w; 2s+1) G_4(y; 2s+1, 2t) H_4(z; 2s+1, 2t),$$
$$\psi_{43}(s,t) \equiv \vartheta_3(x) F(w; 2s+1) G_4(y; 2s+1, 2t+1) H_4(z; 2s+1, 2t+1),$$
$$\psi_{44}(s,t) \equiv \vartheta_2(x) F(w; 2s) G_4(y; 2s, 2t+1)\ H_4(z; 2s, 2t+1),$$

and

$$\Phi_2(x,y,z,w) \equiv \sum_{s=0}^{\alpha'\beta'-1} \sum_{p=0}^{2\sigma'-1} [\psi_{21}(s,p) + \psi_{22}(s,p)],$$

$$\Phi_3(x,y,z,w) \equiv \sum_{s=0}^{\alpha'\beta'-1} \sum_{p=0}^{4\sigma'-1} [\psi_{31}(s,p) + \psi_{32}(s,p)],$$

$$\Phi_4(x,y,z,w) \equiv \sum_{s=0}^{\alpha'\beta'-1} \sum_{t=0}^{2\sigma'-1} [\psi_{41}(s,t) + \psi_{42}(s,t) + \psi_{43}(s,t) + \psi_{44}(s,t)].$$

Let $\Phi_5(x,y,z,w)$, $\Phi_6(x,y,z,w)$ be the functions obtained from $\Phi_3(x,y,z,w)$, $\Phi_4(x,y,z,w)$ respectively by replacing G_3, H_3, G_4, H_4 by G_5, H_5, G_6, H_6 respectively. Then it may be shown as for Φ_1 and S' that

$$\Phi_j(x,y,z,w) = \sum_{\Omega} q^{N(\Omega)} e^{2R(\Omega X')i}, \qquad (j = 1, 2 \cdots 6)$$

where the summation extends over the elements of $S^{(j)}$. For each of the sets $S^{(j)}$ we thus have a Φ_j such that the terms of Φ_j are in one-to-one correspondence with the elements of $S^{(j)}$. By Miss Darkow's results, the sets indicated below have the properties R, C, U, M according as the algebra is of type A, B, C or D.

Type A, S',
Type B, S'',
Type C, S''', S^V,
Type D, S^{IV}, S^{VI}.

If we let S be one of the $S^{(j)}$ which has these properties and let Φ be the corresponding Φ_j, then the argument of Latimer's paper beginning, "Let

x_1, y_1, z_1, w_1 be . . .," and leading to equation (1) may be repeated word for word, except that we set

$$X_1 = x_1 - \frac{y_1}{2\beta'\sigma'} I + \frac{z_1}{2\alpha'\sigma'} J - \frac{w_1}{\alpha'\beta'} K$$

and employ the Φ, S, X, Ω of this paper. We obtain

THEOREM 1. *Let S be one of the $S^{(j)}$ $(j = 1, 2, \cdots 6)$ which has the properties R, C, U, M and let Φ be the corresponding Φ_j. If U is an element of S of norm unity and if x_1, y_1, z_1, w_1 are defined in terms of x, y, z, w by means of $XU = X_1$, then*

$$\Phi(x, y, z, w) = \Phi(x_1, y_1, z_1, w_1).$$

The theorem is trivial for certain U's, for example 1. Furthermore, there may not be a set with a unit yielding a non-trivial identity. It may be shown that if $\alpha = \beta = 1$ and σ be a product of distinct primes in the form $4n + 1$, then the algebra contains a set with a unit which yields a non-trivial identity.

4. *An identity based on $\Delta = s^2 + prt^2$.* The sets of elements used hereafter will not in general be integral. Let r and p be positive integers and let s and t be integers such that s is prime to

$$\Delta = s^2 + prt^2.$$

Let $\rho = r^{\frac{1}{2}}$, $\tau = ip^{\frac{1}{2}}$ and let S be the totality of numbers in the form

$$\Omega = \Delta^{-\frac{1}{2}}(\xi\rho + \eta\tau),$$

where ξ and η are integers such that

$$s\xi \equiv pt\eta \bmod \Delta.$$

Let S_1 be the totality of numbers in the form

$$\Omega_1 = \xi_1\rho + \eta_1\tau,$$

where ξ_1 and η_1 are ordinary integers.

If $U = \Delta^{-\frac{1}{2}}(s + t\rho\tau)$ and

$$\Omega U = \Omega_1$$

it may be shown that Ω_1 belongs to S_1 if and only if Ω belongs to S.

It may be shown that if

$$\Phi_\mu(x, y) \equiv q^{p\mu^2} e^{2p\mu i(rtx+sy)} \vartheta_3[\Delta rx - prt\mu ki, q^{r\Delta}] \vartheta_3[\Delta py - ps\mu ki, q^{p\Delta}],$$

where $k = \log q$, then

$$\Phi_\mu(x, y) = \sum q^{(r\xi^2+p\eta^2)/\Delta} e^{2(r\xi x+p\eta y)i}$$

where ξ and η range over all integers such that

$$\xi \equiv pt\mu \bmod \Delta, \qquad \eta \equiv s\mu \bmod \Delta.$$

Then

$$\Phi(x, y) \equiv \sum_{\mu=0}^{\Delta-1} \Phi_\mu(x, y) = \sum_{\Omega} q^{N(\Omega)} e^{2iR(\Omega X')}$$

where $X = \Delta^{\frac{1}{2}}(\rho x + \tau y)$, and the summation extends over all numbers of S.

It may also be shown that

$$\Psi(x_1, y_1) \equiv \vartheta_3(rx_1, q^r)\vartheta_3(py_1, q^p) = \sum_{\Omega_1} N(\Omega_1) e^{2R(\Omega_1 X'_1)i}$$

where $X_1 = \rho x_1 + \tau y_1$, and the summation extends over all numbers of S_1.

Let x_1, y_1 be defined in terms of x, y by

$$X_1 = XU,$$

or

$$x_1 = sx - pty,$$
$$y_1 = rtx + sy.$$

By means of $\Omega_1 = \Omega U$ we have a one-to-one correspondence between the terms of Φ and Ψ. For corresponding terms

$$N(\Omega_1) = N(\Omega)N(U) = N(\Omega),$$
$$\Omega_1 X'_1 = (\Omega U)(XU)' = (\Omega U)(U'X') = \Omega X'.$$

Hence corresponding terms are equal and

$$\Psi(x_1, y_1) = \Phi(x, y).$$

If we express this in terms of theta functions and replace $r\Delta x$, $p\Delta y$, rx_1, py_1 by x, y, x_1, y_1 respectively, we obtain the identity below.

THEOREM 2. *If r and p are positive integers and s, t are integers such that s is prime to*

$$\Delta = s^2 + prt^2$$

then

$$\vartheta_3(x_1, q^r)\vartheta_3(y_1, q^p) = \sum_{\mu=0}^{\Delta-1} q^{p\mu^2} e^{2\mu i v_1} \vartheta_3(x - prt\mu ki, q^{r\Delta})\vartheta_3(y - ps\mu ki, q^{p\Delta})$$

where $\qquad x_1 = (sx - rty)/\Delta, \qquad y_1 = (ptx + sy)/\Delta.$

5. *An identity based on $\Delta^2 = s^2 + prt^2$.* Let p, r, s, t be defined as before and let

$$\Delta^2 = s^2 + prt^2$$

where Δ is a positive integer prime to s.

Let S be the totality of numbers in the form

$$\Omega = \xi\rho + \eta\tau,$$

where ξ, η are integers such that

$$s\xi \equiv pt\eta \bmod \Delta,$$

and let S_1 be the totality of numbers in the form

$$\Omega_1 = \xi_1\rho + \eta_1\tau$$

where ξ_1, η_1 are integers such that

$$s\xi_1 \equiv -pt\eta_1 \bmod \Delta.$$

Let $$U = s + t\rho\tau.$$

If $\Omega U = \Delta\Omega_1$ then Ω_1 belongs to S_1 if and only if Ω belongs to S. If μ is an arbitrarily chosen integer and if

$$\Phi_\mu(x, y) \equiv q^{p\Delta^2\mu^2} e^{2p\mu i(rtx+sy)} \vartheta_3(r\Delta x - \Delta prt\mu ki, q^{r\Delta^2}) \vartheta_3(p\Delta y - p\Delta s\mu ki, q^{p\Delta^2})$$

then

$$\Phi_\mu(x, y) = \sum q^{(r\xi^2+p\eta^2)} e^{2(r\xi x+p\eta y)i}$$

where the summation extends over all pairs of integers ξ, η such that

$$\xi \equiv pt\mu, \qquad \eta \equiv s\mu \bmod \Delta.$$

Therefore

$$\Phi(x, y) \equiv \sum_{\mu=0}^{\Delta-1} \Phi_\mu(x, y) = \sum_{\Omega} q^{N(\Omega)} e^{2R(\Omega X')i}$$

where the summation extends over all the numbers of S and

$$X = x\rho + y\tau.$$

Let

$$\Psi_\mu(x_1, y_1) \equiv q^{p\Delta^2\mu^2} e^{2p\mu i(sy_1-rtx_1)} \vartheta_3(r\Delta x_1 + \Delta prt\mu ki, q^{r\Delta^2}) \vartheta_3(p\Delta y_1 - p\Delta s\mu ki, q^{p\Delta^2})$$

then

$$\Psi_\mu(x_1, y_1) = \sum q^{(r\xi_1^2+p\eta_1^2)} e^{2(r\xi_1 x_1+p\eta_1 y_1)i}$$

and the summation extends over all pairs of integers ξ_1, η_1 such that

$$\xi_1 \equiv -pt\mu, \qquad \eta_1 \equiv s\mu \bmod \Delta.$$

Therefore

$$\Psi(x_1, y_1) \equiv \sum_{\mu=0}^{\Delta-1} \Psi_\mu(x_1, y_1) = \sum_{\Omega_1} q^{N(\Omega_1)} e^{2R(\Omega_1 X'_1)i}$$

where the summation extends over all numbers of S_1 and

$$X_1 = x_1\rho + y_1\tau.$$

The terms of Φ are in one-to-one correspondence with the terms of Ψ and for corresponding terms $N(\Omega) = N(\Omega_1)$. If we set

$$\Delta X_1 = XU,$$
$$x_1 = (sx - pty)/\Delta,$$
or $$y_1 = (rtx + sy)/\Delta,$$

it follows, as before, that

$$\Phi(x, y) = \Psi(x_1, y_1).$$

If we express this in terms of theta functions and replace q^{Δ^2}, Δrx, Δpy, Δrx_1, Δpy_1 by q, x, y, $-x_1$, y_1 respectively we obtain the identity below.

THEOREM 3. *Let p, r be positive integers and let s, t be integers such that*

$$\Delta^2 = s^2 + prt^2,$$

where Δ is a positive integer prime to s. If

$$F(x, y) \equiv \sum_{\mu=0}^{\Delta-1} q^{p\mu^2} e^{2\mu y_1 i} \vartheta_3(x - prt\mu ki/\Delta, q^r)\vartheta_3(y - ps\mu ki/\Delta, q^p),$$

then

$$F(x, y) = F(x_1, y_1),$$

where

$$x_1 = (-sx + rty)/\Delta, \qquad y_1 = (ptx + sy)/\Delta.$$

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ON THE ZEROS OF A POLYNOMIAL AND OF ITS DERIVATIVE.

By HUBERT E. BRAY.

1. *Introduction.* This paper deals with polynomials all of whose roots (zeros) are real.* We subject the roots of the polynomial $P(x)$ to certain simple operations and study the effect of these operations upon the roots of the derivative $P'(x)$. The main object of this study was to find a proof of the theorem (A), below, which is a striking extension of the classical theorem on the coincidence of the respective centroids of the roots of $P(x)$ and $P'(x)$.

The statement of the theorem, communicated to the writer by Doctor J. Shohat, originated with M. T. Popovici.

THEOREM A. *If $P(x)$ is a real polynomial of degree $n \geq 2$, whose roots $x_1, x_2, \cdots x_n$, are all real and non-negative, and if the roots of $P'(x) = D_x P(x)$ are $\xi_1, \xi_2, \cdots, \xi_{n-1}$ and if m is a positive integer, then*

$$\sum_{i=1}^{n} x_i^m/n \geq \sum_{i=1}^{n-1} \xi_i^m/(n-1).$$

The equality is valid if, and only if, either $m = 1$ or all the roots are equal.

It is a little surprising that a theorem so simple in statement, whose proof requires fairly elementary methods, should be new. This may be due to the fact that the theorem, so far as the writer could determine, is not immediately deducible from the well-known recurrence formulae of Newton, which would seem at first glance to provide a natural method of proof. The writer proved a few special cases of the theorem ($m = 2, 3, 4, 5$), by direct use of Newton's formulae, but the calculations became quite complicated as m increased and seemed to lead to no general method. However, he was led by this means to a stronger inequality than that of Theorem (A). This inequality is established in general in § 4.

2. *Auxiliary Theorems.*

THEOREM 1. *If $P(x)$ is a real polynomial, all of whose roots are real, then in the domain of real numbers the polynomial*

$$P'^2 - PP''$$

* In what follows we shall refer to the roots of $P(x) = 0$ as roots of $P(x)$, instead of zeros; this is in order to avoid awkwardness of expression in distinguishing between zeros which are equal to zero and those which are not.

is never negative and vanishes at multiple roots of $P(x)$, but at no other points. If x' is a multiple root of $P(x)$, of order k, x' is a multiple root of $P'^2 - PP''$ of order $2k-2$.

The proof is immediate. Write

$$P(x)=(x-x')^kQ(x)$$

where $Q(x')\neq 0$. We have, after differentiating twice,

$$P'^2 - PP'' = (x-x')^{2k}(Q'^2 - QQ'') + k(x-x')^{2k-2}Q^2.$$

Assuming that the theorem is true for the polynomial Q of lower degree it follows immediately for P. It is obviously true when Q is of the form $(x-x_0)^r$ and we may take this as the initial case in order to complete the proof by induction.

The proof of theorem (A) may be simplified in the following manner. Let

$$\delta^{(m)} = \sum_{i=1}^{n} x_i^m/n - \sum_{i=1}^{n-1} \xi_i^m/(n-1)$$

and suppose that all the roots of $P(x)$ and $P'(x)$ are subjected to a positive displacement of magnitude h; then the quantity $\delta^{(m)}$ becomes a function of h, $\delta^{(m)}(h)$; $\delta^{(m)}(0)=\delta^{(m)}$. Replacing x_i by x_i+h, ξ_i by ξ_i+h we obtain easily

$$\delta^{(m)}(h) = \delta^{(m)} + \binom{m}{1}\delta^{(m-1)}h + \binom{m}{2}\delta^{(m-2)}h^2 + \cdots + \binom{m}{m-2}\delta^{(2)}h^{m-2},$$

since $\delta^{(1)}$ is zero. It follows immediately, from this formula, that if Theorem A is true in the case where the smallest root of $P(x)$ is zero, then it is true, *a fortiori*, when all the roots are positive. In future, therefore, we may confine our attention to this case and we shall write $P(x)=x^kQ(x)$, where $Q(x)$ is a polynomial of degree $n-k$.

We next consider the effect produced upon the roots of $P'(x)$ when we subject the non-zero roots of $P(x)$ to a positive displacement, the other roots of $P(x)$ remaining equal to zero. To fix the ideas let $Q(x)$ represent a polynomial whose smallest root is equal to zero and write

$$P(x,h)=x^kQ(x-h),$$

so that the smallest positive root of $P(x,h)$ is equal to h. By varying the parameter h the positive roots of $P(x,h)$ are rigidly displaced. We now prove

THEOREM 2. *If*

$$P(x,h)=x^kQ(x-h)$$

where k is a positive integer, and if ξ is a positive root of $P'(x,h) = (\partial/\partial x)P(x,h)$, but not a root of $P(x,h)$, then ξ is an increasing function of h such that

$$0 < d\xi/dh < 1, \qquad (h \geq 0).$$

We have

$$P'(x,h) = kx^{k-1}Q(x-h) + x^kQ'(x-h).$$

Consequently,

$$\xi = -kQ(\xi-h)/Q'(\xi-h).$$

Since ξ is positive, and not a root of $Q(x-h)$, $Q(\xi-h)$ and $Q'(\xi-h)$ are different from zero and of opposite signs. Differentiating with regard to h we have

$$\xi' = \frac{d\xi}{dh} = k\,\frac{Q'^2(\xi-h) - Q(\xi-h)Q''(\xi-h)}{Q'^2(\xi-h)}\,(1-\xi').$$

By Theorem 1 the first factor of the right-hand member is positive, since ξ is not a multiple root of $Q(x-h)$. Hence ξ' and $1-\xi'$ are of the same sign and therefore both are positive. Explicitly we obtain, on eliminating k,

$$\xi' = \xi(Q'^2 - QQ'')/[\xi(Q'^2 - QQ'') - QQ'].$$

This quantity is clearly positive, and less than unity, since QQ' is negative; thus the theorem is proved.

3. *Proof of Theorem A.* The proof of Theorem A is effected by showing that Theorem A is implied by Theorem B, below, as follows: Suppose that the different roots of $P(x)$ in ascending order are $0, a_1, a_2, \cdots, a_p$, with respective multiplicities $m_0, m_1, m_2, \cdots, m_p$. Let

$$a_1 = h_1,\ a_2 - a_1 = h_2, \cdots a_p - a_{p-1} = h_p,$$

where $h_1, \cdots, h_p > 0$, and consider the sequence of polynomials, of degree n;

$$P_0 = P,\ P_1,\ P_2, \cdots P_p,$$

which are related in the following manner:

The roots of P_i are obtained from those of P_{i-1} by decreasing all the positive roots of P_{i-1} by the same quantity h_i, the zero roots of P_{i-1} remaining unaltered, $(i = 1, 2, \cdots, p)$. In p stages $P(x)$ is thus transformed into $P_p(x) = x^n$, the case in which Theorem A is evident. Hence the theorem will be established in general if we can show that the quantity $\delta^{(m)}$ formed for $P_{i-1}(x)$ is greater than the corresponding quantity formed for $P_i(x)$. But if we represent $P_i(x)$ by the formula

$$P_i(x) = x^lQ(x)$$

where the smallest root of $Q(x)$ is equal to zero, then

$$P_{i-1}(x) = x^k Q(x - h_i) \qquad (i = 1, 2, \cdots, p).$$

It will be sufficient, therefore, to prove that if

$$P(x, h) = x^k Q(x - h)$$

then $\delta^{(m)}$ formed for the polynomial $P(x, h)$ is an increasing function of h. In other words, Theorem (A) is implied by the

THEOREM B. *If the smallest root of $Q(x)$ is zero and if $h \geq 0$, and*

$$P(x, h) = x^k Q(x - h) \qquad (k \geq 1)$$

and

$$\delta^{(m)}(h) = \sum_1^n x_i^m/n - \sum_1^{n-1} \xi_i^m/(n-1) \qquad (m \geq 2)$$

where $\{x_i\}$ are the roots of $P(x, h)$ and $\{\xi_i\}$ the roots of $P'(x, h)$, then $\delta^{(m)}(h)$ is an increasing function of h.

We prove that $d\delta^{(m)}(h)/dh$ is positive. On differentiating with regard to h, remembering that dx_i/dh is equal to 1, if x_i is positive, and equal to zero otherwise,

$$d\delta^{(m)}(h)/dh = m[\sum_1^n x_i^{m-1}/n - \sum \xi_i^{m-1}\xi'_i/(n-1)].$$

Hence

$$d\delta^{(m)}(h)/dh - m\delta^{(m-1)}(h) = m \sum_{i=1}^{n-1} \xi_i^{m-1}(1 - \xi'_i)/(n-1).$$

Now, if h is positive at least one of the roots x_i is positive and therefore at least one of the roots ξ_i is positive and unequal to any x_i. Hence, by Theorem 2, the right hand member of this equation is positive. The proof is now completed by induction. Assuming that $\delta^{(m-1)}(h)$ is an increasing function it follows that Theorem (A) is true in this case; that is, $\delta^{(m-1)}(h)$ is non-negative and consequently

$$d\delta^{(m)}(h)/dh > 0 \qquad (h > 0).$$

Since $\delta^{(m)}(h)$ is clearly a continuous function it is an increasing function $(h \geq 0)$.

Hence Theorem B, and with it, Theorem A, is proved.

4. *Further Inequalities.* In this section we shall find it convenient to introduce a slight change of notation. Let $P_1(x)$ represent the polynomial $P(x)/(x - x_1)$ where $x_1 \geq 0$ is the smallest root of $P(x)$. We designate the roots of

$$\begin{aligned} P(x) &\quad \text{by} \quad x_1, x_2, \cdots, x_n, \\ P_1(x) &\quad \text{by} \quad x_2, \cdots, x_n, \\ P'(x) &\quad \text{by} \quad \xi_2, \xi_3, \cdots, \xi_n, \\ P'_1(x) &\quad \text{by} \quad \eta_3, \eta_4, \cdots, \eta_n, \end{aligned}$$

each set of roots in order of increasing magnitude.

Consider the quantity

$$\theta^{(m)} = n\left(\sum_1^n x_i{}^m - \sum_2^n \xi_i{}^m\right) - (n-1)\left(\sum_2^n x_i{}^m - \sum_3^n \eta_i{}^m\right).$$

We shall prove in this section that $\theta^{(m)}$ is positive except in two cases

(a) when all the roots are equal to zero,

(b) when $x_1 = 0$ and $m = 1$.

If we increase all the roots by an amount h we find, after replacing x_i by $x_i + h$, ξ_i by $\xi_i + h$, etc., the new value of $\theta^{(m)}$,

$$\theta^{(m)}(h) = \theta^{(m)} + \binom{m}{1}\theta^{(m-1)}h + \cdots + \binom{m}{m-2}\theta^{(2)}h^{m-2} + \binom{m}{m-1}\theta^{(1)}h^{m-1} + h^m.$$

This formula shows that $\theta^{(m)}$ is increased by a positive displacement of all the roots. Hence we may confine our attention to the case where x_1, the smallest root of $P(x)$, is zero.

THEOREM 3. *If $P(x)$ is of the form $x^kQ(x)$ where k is positive but not necessarily an integer, and $Q(x)$ is a polynomial of degree ≥ 1 whose roots are real and positive; if ξ is a positive root of $P'(x) = D_xP(x)$, but not a root of $P(x)$, then ξ is an increasing function of k.*

We have

$$P'(x) = x^kQ'(x) + kx^{k-1}Q(x).$$

The positive roots of $P'(x)$ are roots of the polynomial

$$xQ'(x) + kQ(x),$$

that is

$$\xi Q'(\xi) + kQ(\xi) = 0. \tag{1}$$

Regarding ξ as a function of the parameter k, we have, on differentiating with regard to k

$$\{(k+1)Q'(\xi) + \xi Q''(\xi)\}\xi' + Q(\xi) = 0$$

and on eliminating k from the last two equations

$$[Q(\xi)Q'(\xi) + \xi\{Q(\xi)Q''(\xi) - Q'^2(\xi)\}]\xi' + Q^2(\xi) = 0$$

or

$$\xi' = d\xi/dk = Q^2/[\xi(Q'^2 - QQ'') - QQ'].$$

From (1) we see that Q, Q' are different from zero and of opposite signs. And since $Q'^2 - QQ''$ is non-negative, by Theorem 1, it follows that ξ' is positive and the theorem is proved.

N. B. If $k = 0$, that is $P(x) \equiv Q(x)$, then it may happen that P' has no positive root which is not a (multiple) root of P. This will be the case if, and only if, P has but one positive root and this a multiple root.

It is worth while also to remark that if ξ is a positive root of $P'(x)$ and also a (multiple) root of $P(x)$ then $d\xi/dk \equiv 0$ since multiplication by a factor x^k will not affect the positions of multiple roots of $P(x)$.

We utilize this theorem in order to show that if k is an integer ≥ 0, and if

$$P(x) = x^{k+1}Q(x),$$
$$P_1(x) = x^kQ(x),$$

where $P(x)$ is a polynomial of degree $n \geq 3$, and $Q(x)$ is the polynomial of Theorem 3, then

$$\sum_2^n \xi_i^m > \sum_3^n \eta_i^m$$

where $\xi_2, \cdots, \xi_n$ are the roots of $P'(x)$ and $\eta_3, \cdots, \eta_n$ those of $P'(x)$, the roots being arranged in ascending order of magnitude. This result follows immediately from Theorem 3 if k is positive, since if ξ_i is a positive root of P' but not a root of P, (such a root always exists if $k > 0$), it is greater than the corresponding root η_i of P_1'. But if $k = 0$ then P' has exactly one positive root ξ_2 lying between 0 and the smallest root of $Q(x)$, i. e. $P_1(x)$; each of the other roots ξ_i of P', $(i \geq 3)$, is equal to or exceeds the corresponding root η_i of P' according as ξ_i is a (multiple) root of P or not, by Theorem 3 and the remark following it.

In any case therefore the inequality

$$\sum_{i=2}^n \xi_i^m > \sum_{i=3}^n \eta_i^m$$

is valid. Introducing the abbreviations

$$p_m = \sum_1^n x_i^m - \sum_2^n \xi_i^m,$$

$$q_m = \sum_2^n x_i^m - \sum_3^n \eta_i^m,$$

we may now state the result:

$$p_m < q_m,$$

which will be used to prove

THEOREM C. *If $Q(x)$ is a polynomial of degree ≥ 1 whose roots are real and positive, and if $P(x)$ is a polynomial of degree $n \geq 3$ of the form:*

$$P(x) = x^{k+1}Q(x),$$

and

$$P_1(x) = x^k Q(x),$$

where k is an integer ≥ 0; then

$$n(\sum_1^n x_i^m - \sum_2^n \xi_i^m) > (n-1)(\sum_2^n x_i^m - \sum_3^n \eta_i^m)$$

where $\xi_i (i = 2, \cdots, n)$, $\eta_i (i = 3, \cdots, n)$, $x_i (i = 1, \cdots, n)$, $x_i (i = 2, \cdots, n)$ are the roots of P', P_1', P, P_1, respectively, and m is an integer ≥ 2.

We first observe that $\sum_1^n x_i^m$ is the coefficient of $x^{-(m+1)}$ in the expansion of the rational function $P'(x)/P(x)$, analytic at ∞, in powers of $1/x$. For if we write

$$\sum_1^n 1/(x - x_i) = P'(x)/P(x) = \sum_{i=0}^{\infty} a_i x^{-i}$$

then it follows that

$$\sum_{i=1}^n x_i^m = (1/2\pi i)\int_C [P'(x)/P(x)]x^m dx = a_{m+1}$$

on integrating termwise about a contour C, in the complex plane, which encloses all the roots of $P(x)$. Consequently the quantity

$$p_m = \sum_{i=1}^n x_i^m - \sum_{i=2}^n \xi_i^m$$

is the coefficient of $x^{-(m+1)}$ in the expansion in powers of $1/x$ of the function

$$P'(x)/P(x) - P''(x)/P'(x) \equiv [(P'^2 - PP'')/P^2]/(P'/P).$$

But since

$$P'/P = \sum_1^n (x - x_i)^{-1}$$

then, also, on differentiating

$$(P'^2 - PP'')/P^2 = \sum_1^n (x - x_i)^{-2}.$$

Thus we have to calculate the coefficient of $x^{-(m+1)}$ in the expansion of

$$\sum_1^n (x - x_i)^{-2} / \sum_1^n (x - x_i)^{-1}.$$

Now this coefficient is the same as the coefficient of y^m in the expansion in powers of y of the function:

$$\sum_1^n (1 - x_i y)^{-2} / \sum_1^n (1 - x_i y)^{-1}$$

obtained by substituting $x = 1/y$ in the preceding and dividing by y. Hence

$$\sum_{i=1}^n (1 - x_i y)^{-2} / \sum_{i=1}^n (1 - x_i y)^{-1} = p_0 + p_1 y + p_2 y^2 + \cdots,$$

or

$$(n + \sum_{r=1}^{\infty} (r+1) s_r y^r) / (n + \sum_{r=1}^{\infty} s_r y^r) = \sum_{r=0}^{\infty} p_r y^r,$$

where $s_r = \sum_{i=1}^n x_i{}^r$. On clearing of fractions and equating coefficients we obtain the recurrence formulae

$$\begin{aligned}
&p_0 = 1 \\
&np_1 + p_0 s_1 = 2s_1 \\
&np_2 + p_1 s_1 + p_0 s_2 = 3s_2 \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&np_m + \sum_1^m s_i p_{m-i} = (m+1) s_m.
\end{aligned}$$

or

$$\begin{aligned}
&np_1 = s_1 \\
&np_2 + s_1 p_1 = 2s_2 \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&np_m + \sum_{i=1}^{m-1} s_i p_{m-i} = m s_m.
\end{aligned}$$

The same analysis provides corresponding recurrence formulae for the quantities $q_m = \sum_{i=2}^n x_i{}^m - \sum_{i=3}^n \eta_i{}^m$. We have immediately, since $x_1 = 0$,

$$(n-1) q_m + \sum_{i=1}^{m-1} s_i q_{m-i} = m s_m.$$

On subtracting the last two equations we have:

$$np_m - (n-1) q_m = \sum_{i=1}^{m-1} s_i (q_{m-i} - p_{m-i}), \qquad (m \geq 2).$$

But we have already proved that $q_i - p_i > 0$ $(i = 1, 2, 3, \cdots)$. Hence

$$\theta^{(m)} = np_m - (n-1) q_m > 0$$

except in the case $m = 1$, where we have $np_1 = (n-1) q_1 = s_1$.

Uniting the result of this theorem with the fact, established at the beginning of this section, that $\theta^{(m)}$ is increased by a rigid displacement of all the roots in the positive direction, we obtain

THEOREM D. *If $P(x)$ is a polynomial of degree $n \geq 3$ whose roots are real and non-negative, and $P_1(x)$ is the polynomial $P(x)/(x - x_1)$, where x_1 is the smallest root of $P(x)$, if we denote the respective roots of P, P_1, P', P_1', by $(x_1, \cdots, x_n)$, $(x_2, \cdots, x_n)$, $(\xi_2, \cdots, \xi_n)$, $(\eta_3, \cdots, \eta_n)$, then the inequality*

$$\theta^{(m)} = n\left(\sum_1^n x_i^m - \sum_2^n \xi_i^m\right) - (n-1)\left(\sum_2^n x_i^m - \sum_3^n \eta_i^m\right) \geq 0$$

is valid; moreover $\theta^{(m)} = 0$ if, and only if, either

(a) $x_1 = 0$ *and* $m = 1$,

or

(b) *all the roots are equal to zero.*

Finally we remark that if $x_1 = 0$, so that $P(x) = xP_1(x)$, then, the hypotheses of Theorem C are satisfied and we have, if $m \geq 2$,

$$\theta^{(m)} = n(n-1)\left\{\frac{\sum_1^n x_i^m}{n} - \frac{\sum_2^n \xi_i^m}{n-1} - \frac{n-2}{n}\left(\frac{\sum_2^n x_i^m}{n-1} - \frac{\sum_3^n \eta_i^m}{n-2}\right)\right\} > 0$$

provided not all the roots are equal to zero.

This formula evidently provides an immediate proof of Theorem A, by recurrence, for all values of $n \geq 3$; for it shows that if Theorem (A) is true for the polynomial $P_1(x)$ of degree $n - 1$, then it is true for the polynomial $P(x)$ of degree n having its smallest root equal to zero; whence the theorem follows, *a fortiori*, when all the roots are positive. When $n = 2$, Theorem (A) may easily be proved by elementary algebra, and when this is done the proof by recurrence is complete.

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ON THE NUMERICAL CALCULATION OF THE CURRENT IN AN ANTENNA.

By F. H. MURRAY.

In a preceding paper * the additional field produced by a system of conductors in an impressed electromagnetic field ($\boldsymbol{E}^0 e^{pt}$, $\boldsymbol{H}^0 e^{pt}$) has been represented in terms of surface integrals, which are equivalent to certain integrals of MacDonald † in the applications to be made. In a calculation which was made of the resonance wavelength of a long wire it appeared that the integral representing $\boldsymbol{H}$ could be taken as the basis of a numerical calculation of the current distribution, giving a fair check with experimental results. The usual formulas of alternating current theory correspond, however, to the representation of $\boldsymbol{E}$, and the problem reduces to the numerical solution of an integro-differential equation. In the present paper the analysis of the problem of a straight wire in an impressed field has been developed and extended to include the theory of the loaded vertical antenna, and the inverted L antenna; the latter is briefly discussed in the special case in which the height is small compared to the length of the antenna.

The theory is based on the method, well-known in the theory of integral equations, of assuming an expansion of the unknown function in a series of mutually orthogonal functions $J_n(s)$ which satisfy the boundary conditions at the open ends of the antenna. If the boundary conditions at the surface of the wire are written down, a set of linear equations can be constructed for the unknown Fourier coefficients of the current. It is shown that in the case of the loaded vertical antenna the functions J_n can be so chosen that the principal diagonal terms in these equations dominate the others; if the antenna wire is of small diameter, a good approximation is obtained with a small number of equations. The assumption often made in practice ‡ that a single current component J_1 gives an adequate approximation can therefore be partially justified analytically. It is shown that when this is possible, the impedance of the loaded antenna may be defined; from this impedance the fundamental period of the antenna and the radiation resistance can be calculated.§

* *American Journal of Mathematics*, Vol. 53, April, 1931, pp. 275-288.

† MacDonald, *Electric Waves* (1903), p. 16. The retarded potentials of Lorentz lead to the same representation after a suitable transformation.

‡ See Stuart Ballantine, "On the Radiation Resistance of a Simple Vertical Antenna at Wavelengths Below the Fundamental," *Proceedings of the Institute of Radio Engineers*, Vol. 12 (1924), p. 823.

§ R. Bechmann, "Zur Theorie der Strahlungskopplung von Kurtzwellen—Anten-

An extension of the analysis to other types of antenna, constructed of straight wires in the same plane, is made possible by the formulas of Appendix IV; the analysis, however, is not developed here.

1. The fundamental equations are taken in the form

$$4\pi \boldsymbol{E}_x = \int_S\int \{[\boldsymbol{n}, \boldsymbol{E}] \operatorname{curl} v' + \boldsymbol{E}_n \operatorname{div} v' + \kappa[\boldsymbol{n}, \boldsymbol{H}]\, v'\}\, dS$$

$$4\pi \boldsymbol{E}_y = \int_S\int \{[\boldsymbol{n}, \boldsymbol{E}] \operatorname{curl} v'' + \boldsymbol{E}_n \operatorname{div} v'' + \kappa[\boldsymbol{n}, \boldsymbol{H}]\, v''\}\, dS$$

$$4\pi \boldsymbol{E}_z = \int_S\int \{[\boldsymbol{n}, \boldsymbol{E}] \operatorname{curl} v''' + \boldsymbol{E}_n \operatorname{div} v''' + \kappa[\boldsymbol{n}, \boldsymbol{H}]\, v'''\}\, dS.$$

It is assumed that the conductivity of the exterior region is zero, while the dielectric constant and the permeability are both unity; hence, $h = \lambda = -\kappa = i\omega/c$. In these formulas

$$\phi = e^{-h\rho}/\rho, \quad \rho = [(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2]^{\frac{1}{2}},$$
$$v' = (\phi, 0, 0), \quad v'' = (0, \phi, 0), \quad v''' = (0, 0, \phi)$$

and the field components appearing in the integrand are those of the total field exterior to the conductors.

If the conductor is a segment of a cylinder, while (r, θ, z) form a right-handed coördinate system on the surface, and (l_0, θ_0, z_0) are the coördinates in space, the axis of the cylinder being the z-axis, then

$$\boldsymbol{n} = (\cos\theta, \sin\theta, 0),$$
$$\boldsymbol{E}_x = \boldsymbol{E}_n \cos\theta - \boldsymbol{E}_\theta \sin\theta, \qquad \boldsymbol{E}_n = \boldsymbol{E}_x \cos\theta + \boldsymbol{E}_y \sin\theta,$$
$$\boldsymbol{E}_y = \boldsymbol{E}_n \sin\theta + \boldsymbol{E}_\theta \cos\theta, \qquad \boldsymbol{E}_\theta = -\boldsymbol{E}_x \sin\theta + \boldsymbol{E}_y \cos\theta.$$

Substituting,

$$[\boldsymbol{n}, \boldsymbol{E}] \operatorname{curl} v' = -\boldsymbol{E}_z \cos\theta \partial\phi/\partial z - \boldsymbol{E}_\theta \partial\phi/\partial y$$
$$[\boldsymbol{n}, \boldsymbol{E}] \operatorname{curl} v'' = -\boldsymbol{E}_z \sin\theta \partial\phi/\partial z + \boldsymbol{E}_\theta \partial\phi/\partial x$$
$$[\boldsymbol{n}, \boldsymbol{E}] \operatorname{curl} v''' = \boldsymbol{E}_z \partial\phi/\partial r.$$

Inserting in the fundamental formulas, if the contributions of the ends can be neglected,

$$4\pi \boldsymbol{E}_{l_0} = \int_S\int \{-\boldsymbol{E}_z \cos(\theta - \theta_0)\partial\phi/\partial z$$
$$(1.1) \qquad + \boldsymbol{E}_\theta/l_0 \partial\phi/\partial\theta_0 - \boldsymbol{E}_r \partial\phi/\partial l_0 + \kappa \boldsymbol{H}_z \sin(\theta - \theta_0)\phi\}\, dS$$

$$4\pi \boldsymbol{E}_{\theta_0} = \int_S\int \{-\boldsymbol{E}_z \sin(\theta - \theta_0)\partial\phi/\partial z$$
$$- \boldsymbol{E}_\theta \partial\phi/\partial l_0 - \boldsymbol{E}_r/l_0 \partial\phi/\partial\theta_0 - \kappa \boldsymbol{H}_z \cos(\theta - \theta_0)\phi\}\, dS$$

$$4\pi \boldsymbol{E}_{z_0} = \int_S\int \{\boldsymbol{E}_z \partial\phi/\partial r + \boldsymbol{E}_r \partial\phi/\partial z + \kappa \boldsymbol{H}_\theta \phi\}\, dS.$$

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nansystemen," *Annalen der Physik*, 5. Folge, 1930, Band 4, Heft 7, p. 829, developed a related method for a different problem.

In cylindrical coördinates,

$$\phi = e^{-ik\rho}/\rho,\ \rho = [(z-z_0)^2 + l_0^2 + r^2 - 2l_0 r\cos\ (\theta-\theta_0)]^{\frac{1}{2}}.$$

If $\boldsymbol{E}$ and $\boldsymbol{H}$ are interchanged, while κ is replaced by λ, the representation of $\boldsymbol{H}$ results. In particular, the average value $\bar{\boldsymbol{H}}$ of $\boldsymbol{H}_{\theta_0}$ on a circle of radius l_0, and the average value of $\boldsymbol{E}_z$ become

$$4\pi\bar{H}(l_0, z_0) = \int_a^b \{-\bar{H}(r,z)r\int_0^{2\pi}(\partial\phi/\partial l_0)\,d\theta_0$$

$$-\lambda\bar{E}(z)r\int_0^{2\pi}\phi\cos\,(\theta-\theta_0)\,d\theta_0\}\,dz \tag{1.2}$$

$$4\pi\bar{E}(l_0, z_0) = \int_a^b \{\bar{E}(z)r\int_0^{2\pi}(\partial\phi/\partial r)\,d\theta_0$$

$$+\bar{\boldsymbol{E}}_r(z)(\partial/\partial z)r\int_0^{2\pi}\phi\,d\theta_0 + \kappa\bar{H}(r,z)r\int_0^{2\pi}\phi\,d\theta_0\}\,dz$$

the integrals with respect to θ_0 being independent of θ. From a statement above it is known that the average components $\bar{\boldsymbol{E}}_z$, $\bar{\boldsymbol{H}}_\theta$ which appear under the integral signs may be replaced by the corresponding averages of the total field components $\bar{\boldsymbol{E}}_z + \bar{\boldsymbol{E}}_z^0$, $\bar{\boldsymbol{H}}_\theta + \bar{\boldsymbol{H}}_\theta^0$. Let

$$K(z-z_0, l_0) = r\int_0^{2\pi}(\partial\phi/\partial l_0)\,d\theta_0, \qquad M(z-z_0, l_0) = r\int_0^{2\pi}\phi\cos\,(\theta-\theta_0)\,d\theta_0.$$

The boundary conditions become

$$\boldsymbol{E}_z + \boldsymbol{E}_z^0 = \boldsymbol{E}'_z, \qquad \boldsymbol{H}_\theta + \boldsymbol{H}_\theta^0 = \boldsymbol{H}'_\theta.$$

For ordinary solid copper or iron conductors of small radius at both low and high frequencies the relation $\boldsymbol{E}'_z = \boldsymbol{C}\boldsymbol{H}'_\theta$ holds very approximately. (Riemann-Weber, II, 2d. Edition, pp. 516-518); at high frequencies

$$C = (1/c)[i\omega\mu/4\pi\sigma\ (\text{e. m. u.})]^{\frac{1}{2}};$$

hence, if

$$\bar{F}(z) = \boldsymbol{E}_z^0(z) - C\bar{\boldsymbol{H}}_\theta^0(z)$$

the first equation may be written

$$4\pi\bar{H}(l_0, z_0) + \int_a^b \{K(z-z_0, r) + \lambda CM(z-z_0, l_0)\}$$

$$\times\bar{H}(r,z)\,dz = \lambda\int_a^b M(z-z_0, l_0)\bar{F}(z)\,dz. \tag{1.3}$$

This equation may be transformed into an integral equation of Fredholm type, but for purposes of calculation it is more convenient in this form.

From the field equation in cylindrical coördinates

$$\lambda(\boldsymbol{E}_r^0 + \boldsymbol{E}_r) = (1/r)\partial(\boldsymbol{H}_z^0 + \boldsymbol{H}_z)/\partial\theta - \partial(\boldsymbol{H}_\theta^0 + \boldsymbol{H}_\theta)/\partial z$$

one obtains by integration

$$\lambda(\bar{\boldsymbol{E}}_r^0 + \bar{\boldsymbol{E}}_r) = -\partial(\bar{\boldsymbol{H}}_\theta^0 + \bar{\boldsymbol{H}}_\theta)/\partial z.$$

In the second equation (1.2) let the components of the total field be substituted under the integral sign; with the aid of the definitions

$$L(z - z_0, l_0) = (1/2\pi)\int_0^{2\pi}(\partial\phi/\partial r)\,d\theta_0, \quad G(z - z_0, l_0) = (1/2\pi)\int_0^{2\pi}\phi\,d\theta_0$$

this equation becomes

$$(1.4)\quad 4\pi\bar{E}(l_0, z_0)/2\pi r = C\int_a^b \bar{H}'(z)L(z - z_0, l_0)\,dz + (d/\lambda dz_0)\int_a^b (d\bar{H}'(z)/dz)G(z - z_0, l_0)\,dz + \kappa\int_a^b \bar{H}'(z)G(z - z_0, l_0)\,dz.$$

An integro-differential equation for $\bar{H}'$ is obtained from the boundary condition along the wire,

$$\lim_{l_0 \to r} \bar{E}(l_0, z_0) = -\bar{E}^0(r, z_0) + C\,\bar{H}'(z_0).$$

From the fundamental equations, the total current through a section of the cylinder perpendicular to the z-axis satisfies the equation $2\pi r H' = 4\pi I$; for a perfect conductor $C = 0$ and the first term on the right of (1.4) vanishes, while at high radio frequencies it is still very small. Neglecting this term, let

$$P(z_0) = -(1/h)\int_a^b (dI/dz)G(z - z_0)\,dz \qquad (-\kappa = \lambda = h)$$

$$W(z_0) = \int_a^b I(z)G(z - z_0)\,dz$$

(1.4) becomes

$$(1.41)\qquad \bar{E}(l_0, z_0) = -dP/dz_0 - hW(z_0) \qquad (l_0 \geqq r).$$

2. If the conductors of an antenna system are not all straight, but are constructed from wire of small radius, it will be assumed that with a negligible error the cross-section of each is circular. Assume also that if $\boldsymbol{H}_\theta$ is the angular tangential component of the total magnetic field $\boldsymbol{H}$, only the average value of $\boldsymbol{H}_\theta$ about a circumference need be considered. (s, θ) may be taken as the coördinates of any point of any conductor, except junctions of several conductors which will be neglected, and approximately $dS = rdsd\theta$ on each conductor.

With these approximations, and neglecting the tangential components of the total electric field as before, the reflected field exterior to the conductors is represented in the form

$$E = -\operatorname{grad} P - h\boldsymbol{W},$$

$$P = (1/4\pi)\int_0^{2\pi} d\theta \int \boldsymbol{E}'_r \phi r ds \qquad \phi = e^{-h\rho}/\rho$$

$$\boldsymbol{W} = (1/4\pi)\int_0^{2\pi} d\theta \int [\boldsymbol{n}, \boldsymbol{H}']\,\phi r ds \quad \rho = [(x_1 - x)^2 + (y - y)^2 + (z_1 - z)^2]^{\frac{1}{2}}.$$

P is the retarded scalar potential, and $\boldsymbol{W}$ the retarded vector potential, if the time-factor $exp(pt)$ is omitted. Neglecting $[\boldsymbol{n}, \boldsymbol{H}']_s$,

$$r\int_0^{2\pi} [\boldsymbol{n}, \boldsymbol{H}']d\theta = r\int_0^{2\pi} \boldsymbol{H}'_\theta d\theta = 4\pi \boldsymbol{I} = 4\pi(idx/ds + jdy/ds + kdz/ds)J.$$

Hence if the variation of $\boldsymbol{H}'_\theta$ over the circumference may be ignored,

$$\boldsymbol{W} = \int \boldsymbol{I} ds \int_0^{2\pi} r\phi d\theta.$$

From the field equations in cylindrical coördinates, (see equations just after (1. 3)) the average field components on the surface of the conductor satisfy the equation

$$h(\bar{\boldsymbol{E}}_r + \bar{\boldsymbol{E}}_r{}^0) = -d(\bar{\boldsymbol{H}}_\theta{}^0 + \bar{\boldsymbol{H}}_\theta)/ds = -d\bar{\boldsymbol{H}}'_\theta/ds = -(2/r)dJ/ds$$

$$\bar{\boldsymbol{E}}'_r = -(2/hr)\,dJ/ds.$$

In the integral for P let $\boldsymbol{E}'_r$ be replaced by its average over a circumference; then if

$$G(s, s_0) = (1/2\pi)^2 \int_0^{2\pi}\int_0^{2\pi} \phi d\theta d\theta_0 = G(s_0, s)$$

at a point (s_0) the averages of P, $\boldsymbol{W}$ become

$$\bar{P}(s_0) = -(1/h)\int G(s, s_0)\,(dJ/ds)\,ds,$$

(2. 1)

$$\bar{\boldsymbol{W}}(s_0) = \int G(s, s_0)\boldsymbol{I}(s)ds, \quad \bar{\boldsymbol{W}}_s(s_0) = \int G(s, s_0)J(s)\cos(s, s_0)ds.$$

The average values being understood, the tangential component of E at the surface of any conductor is represented by

(2. 2) $$\boldsymbol{E}_{s_0} = -dP/ds_0 - h\,\boldsymbol{W}_s(s_0).$$

If the antenna system is composed of several unconnected systems, P and W will be represented by sums of integrals each taken over one connected system.

3. *The vertical antenna.* If the earth is assumed to be perfectly conducting, E_s on the antenna is the same as the E_s due to the current in the antenna and its image in the earth's surface, the earth being absent, and twice the loading impedance connecting the antenna and its image, if the loading impedance is at the base. Let this total impedance be an inductance L in electromagnetic units. The variable current J will be assumed to be represented by an expansion in mutually orthogonal functions J_n which vanish at the open ends of the antenna, and satisfy other conditions to be specified later; on the central inductance it is assumed that each J_n is constant. On the antenna,

$$J = \sum_{n=1}^{\infty} a_n J_n(z),$$

$$\begin{aligned} J_n &= e^{\alpha_n(d+z)} - e^{-\alpha_n(d+z)} && -d \leq z \leq -c \\ &= e^{\alpha_n(d-z)} - e^{-\alpha_n(d-z)} && c \leq z \leq d, \quad d-c = l. \end{aligned} \tag{3.1}$$

The coefficients a_n are to be determined from the boundary conditions along the antenna and the central inductance; on the surface of the coil

$$E_s + E_s^0 = Z'_i J(s)$$

and along the antenna

$$E_z + E_z^0 = Z_i J(z), \qquad Z_i = (1/cr)[\mu i\omega/\pi\sigma]^{\frac{1}{2}}$$

if cZ'_i, cZ_i are the internal self-impedances of the coil and of the antenna, respectively, E_s^0, E_z^0 are the average values of the impressed field.

On the antenna it is assumed that

$$E_z^0 = \sum e_n J_n(z).$$

The constants α_n will be determined later in such a way that the condition of orthogonality on the antenna

$$\left(\int_{-d}^{-c} + \int_{c}^{d}\right) J_m(z) J_n(z) = \begin{matrix} 0, m \neq n \\ N_m, m = n \end{matrix}$$

is satisfied, $N_m = -4l(1 - \sin 2\beta_m l/2\beta_m l)$.

To compute the coefficients a_n, let each side of the boundary equation

$$-dP/ds - hW_s = -E_s^0 + Z_i J \tag{3.2}$$

be multiplied by J_m and the products integrated over the coil and the antenna. Integrating the left-hand side over the coil,

$$-\int_{s_1}^{s_2} J_m(dP/ds_0)\,ds_0 = J_m(c)\,[P(s_1) - P(s_2)]$$

$$\int_{s_1}^{s_2} J_m W_s(s_0)\,ds_0 = J_m(c)J(c)\int_{s_1}^{s_2} ds_0 \int_{s_1}^{s_2} ds\, G(s, s_0)\cos(s_0, s) \tag{3.3}$$

$$= LJ_m(c)J(c)$$

if the axis of the coil is parallel to the antenna. Integrating over the antenna

$$\left(\int_{-d}^{-c} + \int_{c}^{d}\right) J_m(z_0)\,(-dP/dz_0)\,dz_0 = J_m(c)\,[P(c) - P(-c)]$$

$$+ \left(\int_{-d}^{-c} + \int_{c}^{d}\right) P(z_0)\,(dJ_m/dz_0)\,dz_0$$

since J_m vanishes at $-d$ and d. The integral of the left-hand member becomes

$$-hLJ(c)J_m(c) + \left(\int_{-d}^{-c} + \int_{c}^{d}\right)\{P(z_0)\,dJ_m/dz_0 - hW_z(z_0)J_m(z_0)\}dz_0$$

$$= -\Delta E^0 J_m(c) + Z'J(c)J_m(c) + N_m(-e_m + Z_i a_m). \tag{3.4}$$

Assuming

$$\Delta E^0 = \int_{s_1}^{s_2} E_s{}^0 ds, \qquad Z' = \int_{s_1}^{s_2} Z'_i ds$$

writing $\sum A_{mn}a_n$ for the integral on the left of (3.4), this becomes

$$\sum_{n=1}^{\infty}\{A_{mn} - hLJ_m(c)J_n(c) - Z'J_m(c)J_n(c) - \delta_n{}^m N_m Z_i\}a_n$$

$$= -\Delta E^0 J_m(c) - N_m e_m \tag{3.5}$$

$$\delta_n{}^m = \begin{cases} 0, & m \neq n \\ 1, & m = n, \end{cases} \qquad (m = 1, 2, 3, \cdots).$$

Since $h = p/c$, this indicates that the ratio $cA_{mn}/J_m(c)J_n(c)$ has the dimensions of an impedance.

Assume the antenna constant c so small that with a negligible error, $\exp(\alpha_n c) = 1$. Let

$$k_n = h - \alpha_n = i\epsilon_n \mid k_n \mid, \qquad k'_n = h + \alpha_n = i\epsilon'_n \mid k'_n \mid.$$

Then approximately (see Appendix I), if $\gamma = .5772157$,

$$I_n(l) = \int_0^l G(w)e^{\alpha_n w}dw$$

$$= \log(2l/r) - \gamma - \log \mid k_n l \mid + Ci(\mid k_n l \mid) - \epsilon_n iSi(\mid k_n l \mid).$$

Let $h = ik$, and

$$\Delta_n = I_n(2l) - I_n(l).$$

From the analysis of Appendix II it is found that if $\alpha_m + \alpha_n \neq 0$,

$$\begin{aligned} hA_{mn}/2 = &(k^2 - \alpha_m\alpha_n)/(\alpha_m + \alpha_n)\{(I_{-m} + I_{-n})[1 + e^{l(\alpha_m+\alpha_n)}] \\ &-(I_m + I_n)[1 + e^{-l(\alpha_m+\alpha_n)}] + e^{l(\alpha_m-\alpha_n)}(I_{-m} - I_n) \\ &+ e^{-l(\alpha_m-\alpha_n)}(I_{-n} - I_m) - e^{2l\alpha_m}\Delta_{-m} + e^{-2l\alpha_m}\Delta_m - e^{2l\alpha_n}\Delta_{-n} + e^{-2l\alpha_n}\Delta_n\} \\ (3.6)\quad &+ (k^2 + \alpha_m\alpha_n)/(\alpha_m - \alpha_n)\{(I_m + I_{-n})[1 + e^{-l(\alpha_m-\alpha_n)}] \\ &-(I_{-m} + I_n)[1 + e^{l(\alpha_m-\alpha_n)}] + e^{l(\alpha_m+\alpha_n)}(I_{-n} - I_{-m}) \\ &+ e^{-l(\alpha_m+\alpha_n)}(I_m - I_n) + e^{2l\alpha_m}\Delta_{-m} - e^{-2l\alpha_m}\Delta_m + e^{-2l\alpha_n}\Delta_n - e^{2l\alpha_n}\Delta_{-n}\}. \end{aligned}$$

Since the constants α_n are unrestricted above, the coefficients A_{nn} may be calculated by a passage to the limit. With the approximation employed already

$$dI_n(l)/d\alpha_n = (1 - e^{-k_n l})/k_n = I'_n$$
$$dI_{-n}(l)/d\alpha_n = -(1 - e^{-k'_n l})/k'_n = I'_{-n}.$$

Allowing α_m to approach α_n above,

$$\begin{aligned} hA_{nn}/2 = &(k^2 - \alpha_n^2)/\alpha_n\ [I_{-n}(1 + e^{2l\alpha_n}) \\ &-I_n(1 + e^{-2l\alpha_n}) + I_{-n} - I_n + e^{-2l\alpha_n}\Delta_n - e^{2l\alpha_n}\Delta_{-n}] \\ (3.7)\quad &+ (k^2 + \alpha_n^2)\,[-2l(I_{-n} + I_n) + 2(I'_n - I'_{-n}) + I'_n e^{-2l\alpha_n} \\ &- I'_{-n}e^{2l\alpha_n} + 2l(\Delta_{-n}e^{2l\alpha_n} + \Delta_n e^{-2l\alpha_n}) + \Delta'_{-n}e^{2l\alpha_n} - \Delta'_n e^{-2l\alpha_n}]. \end{aligned}$$

For ordinary antenna wire the functions $\log(2l/r)$, $\log(4l/r)$ are the dominant terms in $I_n(l)$ and $I_n(2l)$, respectively; taking only these terms, and replacing Δ_n by zero, it is seen that the corresponding value of A_{mn} becomes

$$16KL' \sin l\beta_m \sin l\beta_n/h,$$

if $\alpha_n = i\beta_n$, and the constants β_n for every n satisfy the equation

$$\beta_n \cot l\beta_n = K, \qquad L' = log\ (2l/r). \tag{3.8}$$

Now $J_n(0) = 2i \sin \beta_n l$, hence the dominant terms on the left of (3.5) become

$$\sin l\beta_m \sin l\beta_n\,[16\,KL' + 4h^2L]/h.$$

These vanish if

$$K = k^2L/4L'$$

except if $m = n$; hence the principal diagonal terms, $m = n$, dominate the others and the equations (3.5) can be solved by the method of successive approximations.

To determine the fundamental period of the loaded antenna, approximately, assume $k = \beta_1$ and consider only the logarithmic terms as before; neglecting Δ_1, Δ_{-1},

$$A_{11} = -16\, i\, L' \cot kl \sin^2 kl$$

and approximately,

$$A_{11} - hL\, J_1(0)^2 = i \sin^2 kl\, [-16\, L' \cot kl + 4kL].$$

This vanishes if

$$\cot kl = kL/4L'.$$

A value of k which satisfies this equation is also one of the set of numbers β_n defined by the transcendental equation (3.8).

If a condenser is in series or in parallel with the inductance, the total retarded potential will have an added term due to its charge; the calculation is similar to the preceding, if the mutual capacity between the condenser plates and any part of the antenna may be neglected.

It is easily verified that the functions J_n determined by equation (3.8) are orthogonal on the antenna.

4. *Theory of the inverted L antenna.* The antenna considered here consists of two arms of equal length l, parallel and with the distance d between the centers of the wires; it is assumed that the length l and the wavelength λ of the fields to be considered are large compared with d, and that the current is sensibly constant on the conductor or conductors joining the two branches. The inverted L antenna over a perfectly conducting earth is equivalent to such a pair of arms, if the earth is assumed absent and the image of the actual antenna is assumed to be present instead.

Let s be the distance increasing from the lower to the upper open end of the antenna, while I has the direction of increasing s at every point, $J(-s) = J(s)$. Employing (2.2) and integrating by parts as before, one obtains the equations (3.5) for the Fourier constants α_n, if

$$hA_{mn} = \int' \int' G(s, s_0)\{k^2 J_m(s_0) J_n(s) \cos(s_0, s) - (dJ_m/ds_0)(dJ_n/ds)\}\, ds ds_0$$

the integration being extended over the antenna alone. The same value of N_m is obtained as before. The interaction between the vertical and horizontal parts of the antenna will be neglected, and L is assumed to be the loading inductance plus the self-inductance of the vertical connecting arm.

Let $h = \alpha_1$, $dt_l = l + [d^2 + l^2]^{\frac{1}{2}}$, and on the upper arm

$$J_m = e^{\alpha_m(l-z)} - e^{-\alpha_m(l-z)}, \quad 0 \leq z \leq l.$$

Let

$$I_1 = \log(2l/r),$$
$$\bar{I}_1 = Ci(dk) - Ci(dk/t_l) - i\,[Si(dk) - Si(dk/t_l)]$$

(4.2)

$$I_{-1} = \log(2l/r) - \gamma - \log(2kl) + Ci(2kl) - iSi(2kl)$$
$$\bar{I}_{-1} = -Ci(kd) + Ci(kdt_l) - i\,[Si(dkt_l) - Si(dk)].$$

Under the above assumptions,

$$A_{11} = (4k^2/h^2)\{(1 + e^{2hl})(I_{-1} - \bar{I}_{-1}) - (1 + e^{-2hl})(I_1 - \bar{I}_1)\}.$$

If $d \ll l$, $kd \ll 1$, then $dt_l \doteq 2l$, $d/t_l \doteq d^2/2l$, and

$$Ci(kd) \doteq \gamma + \log kd, \quad Ci(dk/t_l) \doteq \gamma + \log(d^2/2l)$$
$$I_{-1} - \bar{I}_{-1} \doteq I_1 - \bar{I}_1 \doteq \log(d/r) = L'.$$

Hence

$$R(A_{11}/i) \doteq -16L' \cot kl \sin^2 kl$$
$$= 4L' \cot kl\ J_1^2(0), \quad J_1(0) = 2i \sin kl$$

and the imaginary part of the coefficient of a_1 in (4.1) is small if

$$\cot kl = kL/4L'. \tag{4.3}$$

The equation

$$\beta_n \cot \beta_n l = k^2 L/4L'$$

again defines an orthogonal set of functions $J_n(s)$ on the antenna.

If the interaction of the vertical and horizontal parts of the antenna is to be taken into account, it is necessary to employ the formulas of Appendix IV.

5. *Calculation of radiation resistance.* Employing electromagnetic units, ($E_{em} = cE_{es}$), and writing

$$Z_0 = i\omega L + cZ',$$
$$Z_{mn} = (-cA_{mn} + \delta_n{}^m N_m c Z_i)/J_m(0)J_n(0)$$

equations (3.5) become

$$\sum_{n=1}^{\infty} (Z_{mn} + Z_0) a_n J_n(0) = \Delta E^0 + N_m e_m / J_m(0). \tag{5.1}$$

In many cases a rough approximation to the antenna current can be found by considering only the first equation and neglecting $a_2, a_3, \cdots$. If the antenna is transmitting, e_m is zero, and

$$a_1 = \Delta E^0/(Z_{11} + Z_0)J_1(0), \quad J(0) = a_1 J_1(0).$$

If Z is defined by the equation

$$\Delta E^0 = ZJ(0),$$

then

$$Z = Z_0 + Z_{11}.$$

The orthogonal functions J_n were determined in such a way that the imaginary part of $Z_0 + Z_{11}$ is very nearly zero if k is suitably chosen; if the imaginary part is zero, and the remaining Fourier components of the current are negligible, the antenna behaves as a resistance, of amount $Z_0 + Z_{11}$. The constant $R(cZ_4)$ represents the high-frequency resistance per unit length along the antenna; the radiation resistance is $cR(-A_{11})/J_1(0)^2$. For instance, suppose the unloaded vertical antenna excited by a field in which $k = \pi/2l$; in this case $\alpha_n = n\pi i/2l$, and from (3. 7)

$$A_{11} = -4[I_{-1}(2l) - I_1(2l)].$$

Since $J_1(0) = 2i$,

$$\begin{aligned} -cA_{11}/J_1(0)^2 &= c[I_1(2l) - I_{-1}(2l)] \\ &= c[\gamma + \log(2\pi) - Ci(2\pi) + iSi(2\pi)]. \end{aligned}$$

Hence the reactance of the unloaded antenna vanishes only for a wavelength slightly different from $4l$; it can be shown that if

$$L_0 = 2\log(4l/r) - 1 - [\gamma + \log 2\pi - Ci(2\pi)],$$

and the ratio $4l/r$ is large, the fundamental wavelength is very approximately given by

$$1 - 4l/\lambda = Si(2\pi)/\pi[L_0 - 1 + Si(2\pi)/\pi].$$

As the difference is small, the radiation resistance at resonance is very nearly *

$$c[\gamma + \log 2\pi - Ci(2\pi)].$$

The outward energy flow due to the reflected field is easily expressed in terms of the coefficients a_n and A_{mn}. If Σ is a sphere of large radius enclosing the antenna, assumed isolated in space, the energy crossing Σ per unit time is equal to

$$\begin{aligned} S &= (c/8\pi)\int\!\!\int_{\Sigma} ([\boldsymbol{E}, \bar{\boldsymbol{H}}]_n + [\bar{\boldsymbol{E}}, \boldsymbol{H}]_n)\, dS \\ &= -(c/8\pi)\int\!\!\int_{\Sigma} (\boldsymbol{E}\cdot[\boldsymbol{n}, \bar{\boldsymbol{H}}] + \bar{\boldsymbol{E}}\cdot[\boldsymbol{n}, \boldsymbol{H}])\, dS. \end{aligned}$$

* Ballantine, *l. c.*, p. 828 (15).

The integral is independent of the shape or position of the surface; evaluating it on the antenna, and assuming $[\boldsymbol{n}, \boldsymbol{H}]/4\pi = I/2\pi r$, as in the preceding paragraphs,

$$S = -(c/2)\int (E_s\bar{J} + \bar{E}_s J)\,ds.$$

Let E_s have the expansion in orthogonal functions

$$E_s = \sum_{n=1}^{\infty} b_m J_m(s).$$

Then

$$S = -(c/2)\sum_{m=1}^{\infty} (b_m\bar{a}_m + \bar{b}_m a_m)N_m. \tag{5.2}$$

From (4.1),

$$N_m b_m = \sum_n a_n\{-hLJ_m(0)J_n(0) + A_{mn}\}.$$

Hence if $J_m(0)J_n(0)$ is real,

$$S = -(c/2)\sum_{m,n=1}^{\infty} J_m(0)J_n(0)[\bar{a}_m a_n(-hL) + a_m\bar{a}_n(hL)]$$

$$-(c/2)\sum_{m,n=1}^{\infty}[A_{mn}\bar{a}_m a_n + \bar{A}_{mn}a_m\bar{a}_n].$$

The first sum on the right drops out; if all the current components except the first are neglected,

$$S' = -c\,|\,a_1\,|^2(\bar{A}_{11} + \bar{A}_{11})/2.$$

The current at $s = 0$ is given by $J = a_1 J_1(0)$, hence

$$S' = -cJ^2R(A_{11})/J_1(0)^2 = J^2R_1$$

if

$$R_1 = cR(-A_{11})/J_1(0)^2 \tag{5.3}$$

the value found previously.

6. *Explicit formulas for a perfectly conducting earth.* A set of functions $J_n(s)$ which are orthogonal on the antenna with its image are also orthogonal on the upper part alone, while when this half alone is considered, $N'_m = N_m/2$. Also, if the integration is extended only over the upper half in the calculation of A_{mn}, just half the former value is obtained, as a result of the equation $J_n(-s) = J_n(s)$, $s = 0$ denoting as always the grounding point. If Z'_0 denotes the loading inductance in the upper half, $Z'_0 = (1/2)Z_0$ in equation (5.1), and $Z'_{mn} = (1/2)Z_{mn}$. Hence instead of (5.1) one obtains

$$(6.1) \qquad \sum_{n=1}^{\infty} (Z'_{mn} + Z'_0) a_n J_n(0) = \Delta E^0 + N'_m e_m / J_m(0).$$

The radiation resistance, when this can be defined by means of $J_1(s)$, comes out half its former value; the definition of the numbers α_n, β_n is the same as before. The constants e_m in the equation above are now the Fourier constants of the impressed field on the upper half.

7. *The reciprocal theorem.* Let two antennas be present, while the presence of the earth is neglected. Let E'_s be the electric intensity on antenna 1 which would exist if this antenna were absent; $\boldsymbol{E}'$ is the field impressed by acurrent J'' in antenna 2; similarly, $\boldsymbol{E}''$ is the field impressed along antenna 2 by a current J' in antenna 1. The fields $\boldsymbol{E}'$, $\boldsymbol{E}''$ can be represented in the form

$$\boldsymbol{E}' = -\operatorname{grad} \boldsymbol{P}'' - h\boldsymbol{W}'',$$
$$\boldsymbol{E}'' = -\operatorname{grad} \boldsymbol{P}' - h\boldsymbol{W}'.$$

Here

$$\boldsymbol{P}''_{s_1} = -(1/h) \int_{(2)} (dJ''/ds_2) G(s_1, s_2) ds_2,$$

$$\boldsymbol{W}''_{s_1} = \int_{(2)} J'' G(s_1, s_2) \cos(s_1, s_2) ds_2.$$

It follows that

$$\int_{(1)} \boldsymbol{E}'_s J'(s_1) ds_1 = -\int_{(1)}\int_{(2)} J'(s_1)(d/ds_1)[(-dJ''/h ds_2) G(s_1, s_2)] ds_1 ds_2$$

$$-h \int_{(1)}\int_{(2)} J'(s_1) J''(s_2) \cos(s_1, s_2) G(s_1, s_2) ds_1 ds_2$$

$$= (1/h) \int_{(1)}\int_{(2)} G(s_1, s_2) \{k^2 J' J'' \cos(s_1, s_2)$$

$$-(dJ'/ds_1)(dJ''/ds_2)\} ds_1 ds_2.$$

Hence

$$\int_{(1)} \boldsymbol{E}'_s J'(s_1) ds_1 = \int_{(2)} \boldsymbol{E}''_s J''(s_2) ds_2.$$

Let J' be the *mth* orthogonal function on 1, J'' the *nth* orthogonal function on 2; the integrals above represent Fourier expansion constants on their respective antennas, if the ΔE impressed between the terminals of the loading inductances may be neglected. By definition

$$\int_{(1)} J'^2_m ds_1 = N'_m, \quad \int_{(2)} J''^2_n ds_2 = N''_n,$$

and it may be assumed that on the antenna,

$$E'_s = \sum e'_m J'_m(s_1), \quad e'_m = (1/N'_m) \int_{(1)} E'_s J'_m(s_1)\, ds_1.$$

The integral identity becomes,

$$N'_m e'_{nm} = N''_n e''_{mn}$$

if e'_{nm} is *mth* Fourier constant of the field due to the *nth* orthogonal function of the set defined for antenna 2. The first subscript indicates the origin in each case. The functions on each antenna may be normalized by the condition $N'_m = N''_n = 1$; in this case the simpler equation $e'_{nm} = e''_{mn}$ results.

Appendix I.

The integrals $I_n(l)$ may be calculated as follows. Let δ be a number large compared to r, but such that $\alpha\,\delta$ is very small, $h - \alpha = k = \epsilon i \mid k \mid$.

$$I_a(l) = \int_0^l G(w) e^{aw} dw = \left(\int_0^\delta + \int_\delta^l \right) G e^{aw} dw = I' + I''.$$

$$I' = \int_0^\delta G(w)[1 + \alpha w + \alpha^2 w^2/2! + \cdots]\, dw \doteq \int_0^\delta G(w)\, dw$$

$$= (1/2) \int_{-\delta}^{\delta} G(w)\, dw = I_0(hr) K_0(hr)^* - \int_\delta^\infty e^{-hw} dw/w$$

$$= -\log(hr/2) - \gamma + Ci(d \mid h \mid) + i[\pi/2 - Si(d \mid h \mid)]$$

$$= -\log(r/2) - \log \mid h \mid - \pi i/2 - \gamma + [\gamma + \log \delta \mid h \mid] + i[\pi/2 - Si(\delta \mid h \mid)]$$

$$= \log(2\delta/r) + \delta \mid h \mid (\cdots).$$

$$I'' = -[Ci(\delta \mid k \mid) - Ci(l \mid k \mid)] - \epsilon i [Si(l \mid k \mid) - Si(\delta \mid k \mid)].$$

Neglecting powers of $\delta \mid k \mid$,

$$Ci(\delta \mid k \mid) = \gamma + \log \mid k \mid + \log \delta.$$

$$I_a(l) = \log(2l/r) - \gamma - \log \mid kl \mid + Ci(l \mid k \mid) - \epsilon i Si(l \mid k \mid).$$

Appendix II.

Let $J_n(z)$ on the verticle antenna be defined by

(1) $\quad J_n(z) = 2i \sin \beta_n (d + z), \quad a \le z \le b \qquad (i\beta_n = \alpha_n)$

(2) $\quad = 2i \sin \beta_n (d - z), \quad c \le z \le d \cdot \qquad (l = d - c).$

$\qquad a = -d, \qquad b = -c.$

---

* Integrating first from $-\infty$ to $+\infty$, $4\pi k_0(hd)$ results from No. 917 of the tables of G. A. Campbell, *Bell System Technical Journal*, October, 1928, p. 687, if d is the distance from a point outside the cylinder to a line on the surface. The integration with respect to θ is easily performed using the addition theorem, Watson, *Theory of Bessel Functions*, p. 361 (8).

If $h = ik$, and the lower and upper parts of the antenna are denoted by (1), (2) respectively,

$$hA_{mn} = \sum_{i,j=1}^{2} \int_{(i)} dz_0 \int_{(j)} dz G(z - z_0)\{k^2 J_m(z_0) J_n(z) - (dJ_m/dz_0)(dJ_m/dz)\}.$$

Each term of this sum can be expressed linearly in terms of the integrals

$$I_{rs}^{mn} = \int_{(r)} dz_0 \int_{(s)} dz G(z - z_0) e^{a_m z_0 + a_n z} \qquad (r, s = 1, 2).$$

Integrating by parts, assuming $\alpha_m + \alpha_n \neq 0$,

$$I_{11}^{mn} = \int_a^b dz_0 \int_a^b dz G(z - z_0) e^{a_m z_0 + a_n z} = \int_a^b e^{(a_m + a_n) z_0} dz_0 \int_{a - z_0}^{b - z_0} G(w) e^{a_n w} dw$$

$$= [1/(\alpha_m + \alpha_n)]\{e^{b(a_m + a_n)}[I_{-n}(l) + I_{-m}(l)] - e^{a(a_m + a_n)}[I_m(l) + I_n(l)]\}$$

if

$$I_m(l) = \int_0^l G(w) e^{a_m w} dw, \qquad I_{-m}(l) = \int_0^l G(w) e^{-a_m w} dw.$$

Similarly,

$$I_{22}^{mn} = [1/(\alpha_m + \alpha_n)]\{e^{d(a_m + a_n)}[I_{-m}(l) + I_{-n}(l)] - e^{c(a_m + a_n)}[I_m(l) + I_n(l)]\}.$$

Also

$$I_{21}^{nm} = I_{12}^{mn} = [1/(\alpha_m + \alpha_n)]\,[e^{-(a_m + a_n)c} \int_{2c}^{l+2c} G(w) e^{a_n w} dw$$

$$- e^{-(a_m + a_n)d} \int_{l+2c}^{2l+2c} G(w) e^{a_n w} dw$$

$$+ e^{(a_m + a_n)d} \int_{l+2c}^{2l+2c} G(w) e^{-a_m w} dw - e^{(a_m + a_n)c} \int_{2c}^{l+2c} G(w) e^{-a_m w} dw].$$

It is seen that if $c = 0$,

$$I_{11}^{-m,-n} = I_{22}^{m,n} = e^{l(a_m + a_n)} I_{11}^{mn}, \qquad I_{12}^{-m,-n} = I_{12}^{nm} = I_{21}^{mn}.$$

The formulas (3.6) are obtained on substitution of the integrals above in the expression for A_{mn}, $c = 0$.

Appendix III.

Let the superscripts 1, 2 denote the upper and lower arms, respectively; the integral in (4.1) is the sum of four terms

$$A_{mn} = A_{mn}^{11} + A_{mn}^{12} + A_{mn}^{21} + A_{mn}^{22}$$

the first position above and below denoting z_0. On the lower half $s = -z$, $0 \leq z \leq l$. Hence

$$A^{11}_{mn} = A^{22}_{mn} = (1/h)\int_0^l \int_0^l G(z_0, z)\{k^2 J_m(z_0) J_n(z) - (dJ_m/dz_0)\,(dJ_n/dz)\}dz_0 dz$$

$$A^{12}_{mn} = A^{21}_{nm} = (-1/h)\int_0^l \int_0^l G(z_0, z)\{k^2 J_m(z_0) J_n(z) - (dJ_m/dz_0)\,(dJ_n/dz)\,dz_0 dz.$$

$$\bar{G} = e^{-h\rho}/\rho, \qquad \rho = [(z - z_0)^2 + d^2]^{\frac{1}{2}}.$$

The reduction of the double integrals to single integrals takes place as before; if I^{mn}_{22} has the same meaning as in Appendix II,

$$\begin{aligned} hA^{11}_{mn} &= (k^2 - \alpha_m\alpha_n)\,[e^{-l(a_m+a_n)} I^{mn}_{22} + e^{l(a_m+a_n)} I^{-m,-n}_{22}] \\ &\quad - (k^2 + \alpha_m\alpha_n)\,[e^{-l(a_m-a_n)} I^{m,-n}_{22} + e^{l(a_m-a_n)} I^{-m,n}_{22}] \end{aligned}$$

and

$$\begin{aligned} -hA^{12}_{mn} &= (k^2 - \alpha_m\alpha_n)\,[e^{-l(a_m+a_n)} K^{mn}_{12} + e^{l(a_m+a_n)} K^{-m,-n}_{12}] \\ &\quad - (k^2 + \alpha_m\alpha_n)\,[e^{-l(a_m-a_n)} K^{m,-n}_{12} + e^{l(a_m-a_n)} K^{-m,n}_{12}] \end{aligned}$$

if K^{mn}_{12} is obtained from I^{mn}_{12} by replacing G by $\bar{G}$. If $\bar{I}_n$ is obtained from I_n by the same substitution,

$$K^{mn}_{12} = [1/(\alpha_m + \alpha_n)]\{e^{l(a_m+a_n)}(\bar{I}_{-m} + \bar{I}_{-n}) - (\bar{I}_m + \bar{I}_n)\}.$$

The integral

$$\bar{I}_n = \int_0^l e^{a_n w - h[d^2+w^2]^{\frac{1}{2}}} dw/(d^2 + w^2)^{\frac{1}{2}}$$

is easily evaluated by substituting

$$\begin{aligned} dt &= [d^2 + w^2]^{\frac{1}{2}} + w, & [d^2 + w^2]^{\frac{1}{2}} &= (d/2)\,(t + 1/t) \\ (d/t) &= [d^2 + w^2]^{\frac{1}{2}} - w, & w &= (d/2)\,(t - 1/t). \end{aligned}$$

Let

$$k = (h - \alpha)/2, \qquad k' = (h + \alpha)/2.$$

Then

$$\bar{I}_n = \int_{t_1}^{t_2} e^{-dkt - dk'/t} dt/t = \sum_{n=0}^{\infty} [(-dk')^n/n!\,] \int_{t_1}^{t_2} e^{-dkt} dt/t^{n+1}$$

Integrating by parts

$$\begin{aligned} \int_t^\infty e^{-\gamma t} dt/t^n = e^{-\gamma t}[&1/nt^n - \gamma/n\,(n-1)\,t^{n-1} \\ &+ \gamma^2/n(n-1)\,(n-2)\,t^{n-2} + \cdots + (-\gamma)^{n-1}/n\,!\,t] \\ &+ (-\gamma)^n/n\,! \int_t^\infty e^{-\gamma t} dt/t. \end{aligned}$$

The above expansion may be used if $h + \alpha$ is small; if $h - \alpha$ is small, the integral can be written

$$\bar{I}_n = \int_{1/t_2}^{1/t_1} e^{-dk'u - dk/u} du/u$$

and expanded in the same manner. The second and fourth formulas of (2.4) result from the substitution $h = \pm \alpha_1$.

Appendix IV.

If an antenna is constructed of straight wires in the same plane, the constant A_{mn} may be calculated as a sum of integrals over the same segment, already discussed, and integrals of the form

$$I = \int_{y_1}^{y_2} e^{\beta y} dy \int_{x_1}^{x_2} e^{\alpha x - h\rho} dx/\rho; \qquad \rho = [x^2 + y^2 - 2cxy]^{1/2}$$
$$c = \cos(x, y).$$

In a first approximation only integrals occur for which $h - \alpha$ or $h + \alpha$, and $h - \beta$ or $h + \beta$ are small compared to h. If $h - \alpha$ is small, the transformation $\alpha' = -\alpha$, $x' = -x$, $c' = -c$ reduces the integral to the form in which $h + \alpha'$ is small, and this will be assumed.

Let t be defined by

$$y(1-c)t = \rho + x - cy$$
$$y(1+c)/t = \rho - (x - cy)$$

Then

$$\rho = (y/2)[(1-c)t + (1+c)/t],$$
$$x - cy = (y/2[(1-c)t - (1+c)/t], \quad \partial x/\partial t = \rho/t.$$

Substituting

$$k = (h-\alpha)(1-c)/2, \quad k' = (h+\alpha)(1+c)/2.$$
$$I = \int_{y_1}^{y_2} e^{y(\beta+\alpha c)} J(y)\, dy,$$

if

$$J(y) = \int_{t_1}^{t_2} e^{-kyt - k'y/t} dt/t = \int_{yt_1}^{yt_2} e^{-ku - k'y^2/u} du/u, \quad t_1 = t(x_1, y).$$

One obtains

$$\partial J/\partial y = J' + J'',$$
$$J' = [e^{-kyt-k'y/t}[(1/yt)(\partial yt/\partial y) + 2k'/t]\,]_{t_1}^{t_2}$$
$$J'' = 2kk'y \int_{yt_1}^{yt_2} e^{-ku-k'y^2/u} du/u.$$

neglecting terms in k'^2. Integrating by parts,

$$I = [J(y)e^{y(\beta+\alpha c)}/(\beta+\alpha c)]_{y=y_1}^{y=y_2} - [1/(\beta+\alpha c)] \int_{y_1}^{y_2} e^{y(\beta+\alpha c)}(J' + J'')\, dy.$$

Let

$$f(y) = e^{y(\beta+\alpha c)-kyt-k'y/t}$$

$$R(x) = \int_{y_1}^{y_2} f(y)\,(\partial yt/\partial y)\,dy/yt, \qquad S(x) = \int_{y_1}^{y_2} f(y)\,(\partial yt/\partial y)\,dy/t,$$

$$T(x) = \int_{x_1}^{x_2} f(y)\,dy/t.$$

$$V(y) = 2kk' e^{y(\beta+\alpha c)}[y/(\beta+\alpha c) - 1/(\beta+\alpha c)^2].$$

Then if $\beta + \alpha c \neq 0$,

$$I'' = \int_{y_1}^{y_2} J'' e^{y(\beta+\alpha c)}\,dy = [V(y)\int_{yt_1}^{yt_2} e^{-ku}(du/u)]_{y=y_1}^{y=y_2}$$

$$-[2kk'/(\beta+\alpha c)]\,\{S(x_2) - S(x_1) - [R(x_2) - R(x_1)]/(\beta+\alpha c)\} + k'^2(\cdots)$$

and to terms in k'^2

$$I = [J(y)\,e^{y(\beta+\alpha c)}/(\beta+\alpha c)]_{y_1}^{y_2} - [1/(\beta+\alpha c)]$$
$$\times \{R(x_2) - R(x_1) + 2k'\,[T(x_2) - T(x_1)] + I''\}.$$

The integrals R, S, T can be evaluated by the same method. From the definition of t,

$$(\partial yt/\partial y)/t = 1 - x/\rho.$$

Let the variable u be defined by

$$x(1-c)u = \rho + y - cx \qquad \rho = (x/2)[(1-c)u + (1+c)/u]$$
$$x(1+c)/u = \rho - (y - cx). \quad y - cx = (x/2)[(1-c)u - (1+c)/u]$$
$$\partial y/\partial u = \rho/u.$$

Let

$$\gamma = (1+c)/(1-c), \quad m = (h-\beta)(1-c)/2, \quad m' = (h+\beta)(1+c)/2.$$

It results that

$$y = x(1-c)(u-1)(u+\gamma)/2u, \quad 1/t = (u-1)/(u+\gamma).$$
$$y(\beta+\alpha c) - kyt - k'y/t = x(\alpha+\beta c) - mxu - m'x/u.$$

Substituting,

$$R(x) = e^{x(\alpha+\beta c)} \int_{u_1}^{u_2} e^{-mxu-m'x/u}[-1/u + 2/(u+\gamma)]\,du$$

$$S(x) = e^{x(\alpha+\beta c)} \int_{u_1}^{u_2} e^{-mxu-m'x/u}[x(1-c)(1+\gamma/u^2)/2 - x/u]\,du$$

$$T(x) = [x(1-c)e^{x(\alpha+\beta c)}/2] \int_{u_1}^{u_2} e^{-mxu-m'x/u}$$
$$[1 + (\gamma+1)/\gamma u - 1/u^2 - (\gamma+1)^2/\gamma(u+\gamma)]\,du.$$

For small m' the series expansion of Appendix III can be employed.

ON THE GROUPS GENERATED BY TWO OPERATORS OF ORDERS TWO AND THREE WHOSE PRODUCT IS OF ORDER EIGHT.

By H. R. BRAHANA.

1. *Hypotheses and definition of H.* A group G whose generators, S and T, satisfy the relations:

$$S^3 = T^2 = (ST)^8 = 1, \tag{1}$$

has a commutator subgroup K generated by $\sigma_1 = TS^2TS$ and $\sigma_2 = TSTS^2$ which, since $(ST)^8 = \sigma_2^{-1}\sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_1\ S^2 = 1$, contains S. If K contains T also then $\{S, T\}$ is perfect and hence is isomorphic with some simple group of composite order. The knowledge that a group is perfect reduces very greatly the number of additional facts that must be ascertained before the group can be identified because the number of simple groups of low order is small. We shall be interested here in the cases where G is not perfect.

When G is not perfect, K is of index 2 and contains all operators which are squares in G. $(ST)^4$ written in terms of the σ's and S is $\sigma_2^{-1}\sigma_1\sigma_2 \cdot S$. Then $(ST)^4S^2$ is conjugate to σ_1. Hence * we have the

THEOREM. *If S and T satisfy conditions* (1) *and if in addition the order of TS^2TS is of the form $6k \pm 1$, then the commutator subgroup of $\{S, T\}$ contains a perfect group.*

The conjugates of T_1 which we shall use to denote $(ST)^4$ generate an invariant subgroup H^1 which is in K. The quotient group G/H^1 is of order 2, 6, or 24, its generators satisfying conditions which determine the octahedral group. We shall consider cases where G/H^1 is of order 24. Since this group is octahedral the operator TS^2TS must have an order which is a multiple of 3. If this order were exactly 3 it is easy to prove that the order of $\{S, T\}$ is 24 and that the order of (ST) is 4 instead of 8. The lowest order of σ_1, if (ST) is actually to be of order 8, is 6. We propose to consider this case. We are then dealing with groups generated by two operators satisfying (1), isomorphic with the octahedral group, in which TS^2TS is of order 6.

Under the conditions above the group $\{S, (ST)^4\}$ is generated by two operators of orders 3 and 2 whose product is of order 6. Then $\{S, T_1\}$ con-

* "Certain Perfect Groups, etc.," *American Journal of Mathematics*, Vol. 50 (1928), p. 347.

tains an invariant Abelian subgroup * of one-sixth its order, its commutator subgroup, which is generated by one or two operators. This Abelian group is generated by (T_1T_2) and (T_1T_3) where $T_2 = S^{-1}T_1S$ and $T^3 = S^{-1}T_2S$.

The group $\{T_1, T_2, T_3\}$ is generalized dihedral and may or may not be invariant under G, depending on whether or not T transforms T_3 into some operator of $\{T_1, T_2, T_3\}$. If $\{T_1, T_2, T_3\}$ is invariant then the quotient group corresponding to it is of order 24. If it is not invariant then it is one of a set of conjugate subgroups which are transformed according to a group generated by operators satisfying the same conditions as generators of G.

In case $\{T_1, T_2, T_3\}$ is invariant we have the abelian invariant subgroup $\{T_1T_2, T_1T_3\}$ which we denote by H. Then none of the operators S, T, and T_1 is permutable with every operator of the Abelian group and hence $\{S, T\}$ corresponds to a subgroup of order 48 in the group of isomorphisms of H. This subgroup is generated by operators S_i and T_i which satisfy (1) and for which $(S_iT_i)^4$ is an invariant operator. H cannot be cyclic since its group of isomorphisms contains G_{48}. This last fact requires that $\{S, T_1\}$ be among those groups described by Miller (*loc. cit.*) which have non-cyclic commutator subgroups. A further condition on H is imposed by the fact that its group of isomorphisms contains G_{48}. The Abelian groups of order p^2 and type 1, 1 are of particular interest because there is much known of their groups of isomorphisms that is applicable here.

2. *H of order p^2.* Let us suppose that H is of order p^2 and type 1, 1. There is a $1-1$ correspondence between the operators of the group of isomorphisms of H and matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ whose determinants are different from zero, mod p. Let us arrange the operators of I in cosets with respect to the central C which consists of operators of the form $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ and let us arrange the operators of the central C so that the first two are $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. The determinants of matrices in C are all squares. The matrices whose determinants are squares form a group whose order is half the order of I. We may then require that the first half of the cosets contain only operators whose determinants are squares. We may arrange the matrices in each coset of this first half so that the first is of determinant unity for if it were another square, b^2, the set would contain a matrix whose determinant $b^2\alpha_i^2$ for some α_i would be unity. Then the second matrix is also of deter-

* Miller, "On the Groups Generated, etc.," *Quarterly Journal*, Vol. 33 (1901), p. 76.

minant unity. There are no matrices of determinant unity except those of the first two rows of the first half of the cosets. The remaining operators of I may be obtained by multiplying the matrices whose determinants are squares by $\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$ where α is an arbitrary not-square. The group made up of the matrices of the first two rows of the first half of the cosets is $SLH(2, p)$; its quotient group with respect to the two operators in C is $LF(2, p)$.

Now G transforms H according to a group of order 48 which has an invariant operator of order 2. Hence G_{48} will be isomorphic with a subgroup of order 24 of the quotient group of I with respect to C. Since both I/C and the G_{24} contain a subgroup of half its order then the G_{12} of G_{24} is in the invariant subgroup composed of operators which correspond to squares in I/C. G_{48} determines therefore a G_{12} in $LF(2, p)$. The G_{48} can determine a G_{24} in $LF(2, p)$ only if p is of the form $8h \pm 1$.

We require not only that G determine a G_{24} in I/C but also that G be generated by S and T of orders 3 and 2 respectively. Then the determinant of the matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ corresponding to T must be 1 or -1. If this determinant were 1, T could not be of order 2; T might correspond to an operator of order 2 of $LF(2, p)$ but would itself be of order 4.* If G_{24} is in $LF(2, p)$ p is of the form $8h \pm 1$; then since the determinant of T is -1, -1 must be a square, mod p, which requires that p be of the form $4k \pm 1$. Hence,

If G is isomorphic with a G_{24} of $LF(2, p)$ then p is of the form $8h + 1$.

If on the other hand the G_{24} of I/C is not in $LF(2, p)$ then -1 must be a not-square, mod p, and p must be of the form $4k - 1$. Therefore,

There exists no group G of the type in question containing an invariant subgroup H of order p^2 and type 1, 1 where p is of the form $8h - 3$.

Let us consider next an operator S of order 3 which we denote by $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ whose determinant is Δ. Then $\Delta^3 \equiv 1$, mod p, and we have the following conditions* on α, β, γ, and δ:

$$\begin{aligned} \alpha^3 + 2\alpha\beta\gamma + \beta\gamma\delta &\equiv 1, \\ \alpha\beta\gamma + 2\beta\gamma\delta + \delta^3 &\equiv 1, \\ \beta(\alpha^2 + \alpha\delta + \delta^2 + \beta\gamma) &\equiv 0, \\ \gamma(\alpha^2 + \alpha\delta + \delta^2 + \beta\gamma) &\equiv 0. \end{aligned}$$

The last two require that $(\alpha + \delta)^2 - \Delta \equiv 0$, unless both β and γ are zero.

* Klein-Fricke, *Theorie der elliptischen Modulfunktionen*, p. 393.

* The congruences that follow are all mod p.

We have seen that the determinant of T must be -1; then since (ST) must be of order 8 and such that $(ST)^2$ corresponds to an operator of order 2 in $LF(2, p)$ its determinant $(-\Delta)^2$ must be 1. Since we have both $\Delta^2 \equiv 1$ and $\Delta^3 \equiv 1$, it follows that $\Delta \equiv 1$. So whenever not both β and γ are zero we have $\alpha + \delta \equiv \pm 1$. If we select the positive sign the above conditions on α, β, γ, and δ become:

$$\alpha^3 + \alpha\beta\gamma + \beta\gamma(\alpha + \delta) \equiv 1,$$
$$\alpha\delta - \beta\gamma \equiv 1, \qquad \alpha + \delta \equiv 1.$$

The first reduces as follows, in view of the other two:

$$\alpha^3 + \alpha\beta\gamma + \beta\gamma \equiv 1, \qquad \alpha^3 + (\alpha + 1)(\alpha\delta - 1) \equiv 1,$$

$$\alpha^3 + \alpha^2\delta + \alpha\delta - \alpha - 1 \equiv 1, \qquad \alpha^2 + \alpha\delta - \alpha - 1 \equiv 1, \qquad \alpha - \alpha - 1 \equiv 1,$$

which is impossible when $p \neq 2$. Hence we may assume that $\alpha + \delta \equiv -1$ when not both β and γ are zero. Since all the subgroups of order 3 of I/C are conjugate and since for every p there exist operators of order 3 for which not both β and γ are zero, it will not be necessary to consider that alternative. We may therefore assume that our G_{48} contains the particular $S_i = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ We next seek a T_i such that $\{S_i, T_i\}$ is a G_{48} in I.

Let T_i be $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ where α, β, γ, and δ satisfy the relations

$$\alpha + \delta \equiv 0, \qquad \alpha\delta - \beta\gamma \equiv -1,$$

which we determined above. We now consider (S_iT_i) and arrive at further conditions on α, β, γ, and δ

$$(S_iT_i) = \begin{pmatrix} -\gamma & -\delta \\ \alpha - \gamma & \beta - \delta \end{pmatrix}.$$

The determinant of (S_iT_i) is -1.

$$(S_iT_i)^2 = \begin{pmatrix} \gamma^2 - \alpha\delta + \gamma\delta & \gamma\delta - \beta\delta + \delta^2 \\ -\alpha\gamma + \gamma^2 + (\alpha - \gamma)(\beta - \delta) & -\alpha\delta + \gamma\delta + (\beta - \delta)^2 \end{pmatrix} = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}.$$

If this is to be an operator of order 4 and to correspond to an operator of order 2 in $LF(2, p)$ we must have $\alpha' + \delta' \equiv 0$. This condition, in terms of α, β, γ, and δ is

$$\gamma^2 - 2\alpha\delta + 2\gamma\delta + \beta^2 - 2\beta\delta + \delta^2 \equiv 0,$$

which may be written as

$$\gamma^2 + \beta^2 + \delta^2 - 2\beta\delta + 2\gamma\delta - 2\beta\gamma - 2(\alpha\delta - \beta\gamma) \equiv 0,$$

which is

$$(\beta - \gamma - \delta)^2 \equiv 2(\alpha\delta - \beta\gamma) \equiv -2.$$

This last relation states among other things that -2 is a square; this implies that 2 and -1 are simultaneously squares or not-squares. Now -1 is a square if and only if p is of the form $4h+1$, and 2 is a square if and only if p is of the form $8k \pm 1$. Hence, the last condition can be satisfied if p is of the form $8h+1$ or $8h+3$.

Conversely, when p is of the form $8h+1$ or $8h+3$, then -2 is a square, and I contains the operator $T_i = \begin{pmatrix} 1 & -1 \pm (-2)^{\frac{1}{2}} \\ 0 & -1 \end{pmatrix}$ of order 2. I also contains $S_i = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$. (S_iT_i) is $\begin{pmatrix} 0 & 1 \\ 1 & \pm(-2)^{\frac{1}{2}} \end{pmatrix}$ and $(S_iT_i)^4 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ which is of order two and invariant. Therefore S_i and T_i satisfy the relations $S_i^3 = T_i^2 = (S_iT_i)^8 = 1$. Hence, $\{S_i, T_i\}$ is a subgroup of order 48 of the group of isomorphisms of H. We thus have the

THEOREM. *A necessary and sufficient condition that an Abelian group of order p^2 and type 1, 1 have in its group of isomorphisms a group of order 48 isomorphic with the octahedral group and generated by an operator of order two and one of order three is that p be of the form $8h+1$ or $8h+3$.*

The existence of the G_{48} establishes the existence of a group of order $48p^2$. We have now to determine whether this group of order $48p^2$ can be generated by two operators of orders two and three, respectively.

Let us consider G to be written in cosets with respect to H. Then these cosets will correspond to operators of G_{48}, and we shall designate the coset which corresponds to S_i or T_i by H_{S_i} or H_{T_i}. The operators of H_{T_i} are all of even order since T_i is of order 2, and T_i is an operator of the quotient group. If we take any such operator, the cyclic group generated by it is contained half in H and half in H_{T_i}. Since p is an odd prime, *the operator of order two is in H_{T_i}.* Denote one such operator of order two by T.

Likewise, if $p \neq 3$, then H_{S_i} *contains an operator of order three.* For any operator of H_{S_i} has an order a multiple of 3 and generates a cyclic group which lies in H, H_{S_i}, and $H_{S_i^2}$; the operators of order 3 could not be in H. If $p = 3$ the group of isomorphisms of H is G_{48} and it is known that the G of order $48 \cdot 3^2$ can be generated by two operators of orders two and three respectively.* Let us denote one such operator by S. Then ST must be of order 8 because it appears in $H_{S_iT_i}$ and every operator in $H_{S_iT_i}$ has its fourth power in $H_{(S_iT_i)^4}$. All the operators in $H_{(S_iT_i)^4}$ are of order two because one of them must be of order two and the group generated by this operator and H

* Bisshopp, *Bulletin of The American Mathematical Society*, Vol. 37 (1931), p. 94.

is generalized dihedral. Thus we have shown so far that for p of the form $8h+1$ or $8h+3$ there exists a group $\{S, T\}$ contained in G and that S and T satisfy (1). We have yet to show that there exists an $\{S, T\}$ satisfying the same conditions and containing H.

If $\{S, T\}$ contains one operator of H it contains H because the invariant subgroup of index 48 is non-cyclic. So we shall assume that $\{S, T\}$ is a proper subgroup of G. It must then be G_{48}. Then $T_1 = (ST)^4$ will be invariant in $\{S, T\}$. Now since $TM_2T = M_2^{-1}$ the operator M_2T which is in G will be of order 2. Let us denote this operator by T' and consider the group $\{S, T'\}$. If we use the relations

$$\begin{aligned} TM_1T &= M_1M_2^k \quad \text{and} \quad & S^{-1}M_1S &= M_2^{-1} \\ TM_2T &= M_2^{-1} & S^{-1}M_2S &= M_1M_2^{-1}, \end{aligned}$$

where $k = -1 \pm (-2)^{\frac{1}{2}}$, to simplify the relation

$$(ST')^4 = SM_2TSM_2TSM_2TSM_2T$$

we arrive finally at

$$T_1' = (ST')^4 = (ST)^4 \cdot M_1^{-(k^2+3k+4)} \cdot M_2^{-(k^3+4k^2+8k+7)}.$$

On substituting the value of k we get

$$T'_1 = (ST)^4 M_1^{\mp(-2)^{\frac{1}{2}}} \cdot M_2^{\mp(-2)^{\frac{1}{2}}} = T_1 \cdot M_1^{\mp(-2)^{\frac{1}{2}}} \cdot M^{\mp(-2)^{\frac{1}{2}}}.$$

Then if we let $T'_2 = S^{-1}T'_1S$, we have $T'_1T'_2 = M_2^{\pm 3(-2)^{\frac{1}{2}}}$. Since this last operator is not the identity, the group $\{S, T'\}$ must be G. Therefore,

For every prime of the form $8h+1$ *or* $8h+3$ *there exists a group* $\{S, T\}$ *which contains the Abelian group of order* p^2 *and type* 1, 1 *invariantly and transforms its operator according to the group of order* 48 *whose generators as well as* S *and* T *satisfy* (1) $S^3 = T^2 = (ST)^8 = 1$.

3. *H of order* p^n. Now let us suppose that H is of order p^n where $n > 2$. H will have two generators since its group of isomorphisms I contains G_{48}. Then H must contain a characteristic subgroup H_1 of order p^2 and type 1, 1. This subgroup H_1 will be transformed according to a group isomorphic with G_{48}; moreover the invariant operator of G_{48} is the operator which transforms every operator of H into its inverse and therefore transforms every operator of H_1 into its inverse. Hence H_1 must be transformed according to a group of order 48, 24, or 6, which is non-cyclic and contains an invariant operator of order two. Since, moreover, this group is generated by operators of orders 2 and 3 whose product is of order 8, it must be G_{48}. Therefore, the group

of isomorphisms I_1 of H_1 must contain G_{48}. This fact combined with the theorem of § 2 gives the

THEOREM. *If H is of order p^n, then p must be of the form $8h+1$ or $8h+3$.*

The G_{48} of I_1 is isomorphic with some G_{24} in the group $LF(2, p)$ which is determined by H_1. Since these G_{24}'s are all conjugate we may fix attention on one of them. If the invariants of H are not equal then H will contain a characteristic subgroup of order p, which will be in H_1. The subgroups of order p of H_1 may be designated by matrices $(1\ \alpha)$ determined by generators $M_1M_2^{\alpha}$ where M_1 and M_2 are generators of H_1. Then taking the S_i and T_i of § 2 we seek conditions on p that there exist a subgroup of order p in H_1 which is invariant under G_{48}.*

The subgroup $(1\ \alpha)$ is transformed by S_i into $(\alpha\ \ -(1+\alpha))$. If this subgroup were invariant under S_i we should have the congruence

$$\alpha^2+\alpha+1\equiv 0, \text{ mod } p. \tag{1}$$

The operator T_i transforms the subgroup $(1\ \alpha)$ into $(1\ \ -1\pm(-2)^{\frac{1}{2}}-\alpha)$. This is $(1\ \alpha)$ only if $\alpha\equiv -1\pm(-2)^{\frac{1}{2}}-\alpha$, or

$$\alpha\equiv(-1\pm(-2)^{\frac{1}{2}})2, \text{ mod } p. \tag{2}$$

Substituting the value of α from (2) in (1) we get $\frac{1}{4}\equiv 0$, mod p, which is impossible.

It follows therefore that G_{48} leaves no subgroup of order p of H invariant and that H can have no characteristic subgroup of order p. Hence,

If H is of order p^n, p must be of the form $8h+1$ or $8h+3$, n must be an even number $2m$, and H must be of type m, m.

We inquire whether the above necessary conditions on H are sufficient to insure the existence of a group G of order $48p^{2m}$, and of the type in question.

If H is abelian of order p^{2m} and type m, m its group of isomorphisms I is of order $p^{4m-3}(p-1)(p^2-1)$.† I transforms the characteristic subgroup H_1 of order p^2 and type 1, 1 according to its group of isomorphisms I_1 of order $p(p-1)(p^2-1)$, for there is a $1-1$ correspondence between operators of I_1 and matrices $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ of non-zero determinants whose elements are

* The subgroup generated by M_2, whose matrix would be $(0\ \alpha)$ is not considered in this connection because S_i transforms it into another subgroup.

† Ranum, "Groups of Classes of Congruent Matrices, etc.," *Transactions of the American Mathematical Society*, Vol. 8 (1907), p. 87.

integers reduced modulo p, and every such matrix determines an operator of I, viz.

$$M_1 \sim M_1^{\alpha} M_2^{\beta}, \qquad M_2 \sim M_1^{\gamma} M_2^{\delta}$$

where M_1 and M_2 are generators of H. I is therefore isomorphic with I_1 and the corresponding invariant subgroup of H is of order p^{4m-4}. It consists of all matrices $\begin{pmatrix} 1+a & b \\ c & 1+d \end{pmatrix}$ where a, b, c, and d are residues mod p^m which are congruent to zero, mod p.

When p is of the form $8h+1$ or $8h+3$, I_1 contains a G_{48} and I contains a group isomorphic with it. We shall show that I contains a group *simply* isomorphic with it.

I contains the operators $S_i = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ and $T_i = \begin{pmatrix} 1 & -1+(-2)^{\frac{1}{2}} \\ 0 & -1 \end{pmatrix}$ * where the elements of the matrices are residues mod p^m, for -2 is a square mod p^m whenever it is a square mod p.† These operators are of orders 3 and 2 respectively and $(S_iT_i)^4 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. The group $\{S_i, T_i\}$ must then be G_{48}, and we have accordingly established the existence of a group G of order $48p^{2m}$ which contains H invariantly and transforms it according to G_{48}. We have yet to show that G may be generated by operators S and T which satisfy conditions (1) of § 1.

Let the operators of G be arranged in cosets with respect to H, and let H_{Q_i} denote the coset which corresponds to the operator Q_i of the G_{48} in I. The operators of $H(S_iT_i)^4$ are all of even order and one of them must be of order 2 since H contains no operator of even order. Then it follows that all the operators of $H_{(S_iT_i)^4}$ are of order 2. From this it follows that all the operators of $H_{(S_iT_i)}$ are of order 8, and all the operators of $H_{(S_iT_i)^2}$ are of order 4. The operators of H_{T_i} are all of even order and therefore one of them is of order 2. Let this operator of order 2 in H_{T_i} be denoted by T. If $p \neq 3$, then H_{Si} contains an operator of order 3. The argument of § 2 in which the M_1 and M_2 are now taken to be the operators of order p^m which generate H proves that G *of order* $48p^{2m}$ is generated by S and T of orders 3 and 2 whose product is of order 8.

If $p = 3$, then it is conceivable that every operator of H_{S_i} has for a third power some operator not the identity of H. Such an operator must of course be invariant under S_i. Every operator of H may be written as $M_1^{\alpha} M_2^{\beta}$, and this is transformed by S_i into $M_1^{\beta} M_2^{-2\alpha}$. If $M_1^{\alpha} M_2^{\beta}$ is invariant under

* We shall hereafter denote the corresponding operators of I by S_i, and T_i.

† Tchebichef, *loc. cit.*, p. 93.

S_i we must have $\beta \equiv \alpha$ and $\beta \equiv -2\alpha$, mod p^m. From this it follows that $3\alpha \equiv 0$, mod 3^m, and α is a multiple of 3^{m-1}. Hence the only operators of H invariant under S_i are of order 3 and the operators of H_{S_i} must be of orders 3 or 9. Moreover, since $(M_1^{\alpha}M_2^{\beta}S)^3 = S^3$, where S is any operator of H_{S_i}, it follows that every operator of H_{S_i} is of order 3 or every operator is of order 9. We shall show that they cannot be of order 9.

The group G of order $48p^{2m}$ contains an invariant subgroup of order $24p^{2m}$, which in turn contains an invariant subgroup of order $8p^{2m}$; these subgroups consist of operators in the cosets corresponding to the group $\{S_i,(S_iT_i)^2\}$ and the group generated by the conjugates of $(S_iT_i)^2$, respectively. The former group is generated by an operator S of H_{S_i} and the second group. By a theorem proved in *On the groups which contain etc.** the third power of S must be in the central of the group corresponding to $\{S_i,(S_iT_i)^2\}$. We have already seen that S^3 must be $M_1^{\alpha}M_2^{\alpha}$. $(S_iT_i)^2 = \begin{pmatrix} (-2)^{\frac{1}{2}} & -1 \\ 1 & (-2)^{\frac{1}{2}} \end{pmatrix}$, which transforms $M_1^{\alpha}M_2^{\alpha}$ into $M_1^{\alpha(1+(-2)^{\frac{1}{2}})}M_2^{\alpha(-1+(-2)^{\frac{1}{2}})}$. Therefore, S must be of order 3.

Incidentally we have proved the

THEOREM. *The central of G of order $48p^{2m}$ is the identity.*

We are now assured of the existence in G of S and T which satisfy the relations $S^3 = T^2 = (ST)^8 = 1$. We have still to show that S and T may be chosen so that $\{S, T\}$ is G. This will follow provided we can show that $\{S, T\}$ contains one operator of highest order of H. We suppose that $T_1'T_2' = (S'T)^4 \cdot S'^{-1}(S'T)^4S = R$, where S' is in H_{S_i} and R is some operator not of highest order in H. Then if we take $S = M_2S'$ and find $T_1T_2 = (ST)^4 \cdot S^{-1}(ST)^4S$ we get $T_1T_2 = M_1^{-2}M_2^{4-3(-2)^{\frac{1}{2}}} \cdot R$. Since -2 is prime to p and R is not of highest order, T_1T_2 is of highest order. Hence, G is generated by S and T. We have thus arrived at the

THEOREM. *If p is any prime of the form $8h+1$ or $8h+3$ and m is any positive integer, there exists a group of order $48p^{2m}$ which contains invariantly the abelian group of order p^{2m} and type m, m and transforms its operators according to the G_{48} isomorphic with the octahedral group, and G is generated by operators of orders 2 and 3 whose product is of order 8.*

Conversely,

Any group $\{S, T\}$ generated by two operators which satisfy the relations

* *American Journal of Mathematics*, Vol. 52 (1930), p. 915.

$S^3 = T^2 = (ST)^8 = 1$, *which is also isomorphic with the octahedral group, whose commutator* TS^2TS *is of order* 6, *and which contains an abelian invariant subgroup of index* 48 *of two invariants, and of order a power of a prime, is such that* H *is of order* p^{2m} *and type* m, m *and* p *is of the form* $8h + 1$ *or* $8h + 3$.

4. *H of any order.* Now let us suppose the order of H is $p_1^{n_1} \cdot p_2^{n_2} \cdots p_k^{n_k}$ where the p's are distinct primes. The group of isomorphisms of H is the direct product of the groups of isomorphisms of its Sylow subgroups of orders $p_1^{n_1}, p_2^{n_2}, \cdots p_k^{n_3}$. Since I contains G_{48} which is not a direct product some Sylow subgroup of order $p_i^{n_i}$ must have G_{48} in its group of isomorphisms. Hence some p_i is of the form $8h + 1$ or $8h + 3$ and n_i is an even number. Every Sylow subgroup of order $p_i^{n_i}$ is transformed according to a group Γ_i isomorphic with G_{48} and, unless all the operators of the Sylow subgroup are of order 2, Γ_i contains an invariant operator of order 2. This operator, which transforms every operator of H into its inverse, is the fourth power of an operator in Γ_i and hence Γ_i contains operators of order 8. Therefore Γ_i is G_{48} itself. From this it follows that every one of the p's which is different from 2 is of the form $8h + 1$ or $8h + 3$, and the n's are all even numbers. If p_1 were 2 the operator $(ST)^4$ which transforms every operator of H into its inverse might transform every operator of the Sylow subgroup of order $p_1^{n_1}$ into itself, in which case the Sylow subgroup would be of type $1, 1, 1, \cdots$ and hence would be of order 4. If the Sylow subgroup is not of type 1, 1 then Γ_1 is G_{48}. Moreover, the operators of order 4 would be transformed according to G_{48}. This requires that both invariants be at least as great as 2, and that the Sylow subgroup contain a characteristic subgroup of order 16 and type 2, 2 whose group of isomorphisms contains G_{48}. This is impossible.*

Let us consider the possibility of H being the four-group. Then T_1T_2 is of order 2. The group $\{M_1, M_2, T_1\}$ is abelian of order 8 and type 1, 1, 1 and is invariant. It is transformed according to a group of order 24 in its group of isomorphisms, or a group of lower order isomorphic with it. The group will then be in the holomorph of G_8, and can be written on 8 letters. The operators of G_8 are transformed according to an intransitive group on seven letters, since $\{M_1, M_2\}$ is invariant. The four operators of G_8 not in $\{M_1, M_2\}$ must then be transformed according to G_{14}. These four operators must contain the fourth powers of all the operators of order 8 in $\{S, T\}$, and

* The group of isomorphisms of the group of order 16 can be written on 12 letters for the operators of order 4. The fourth power of an operator of order 8 would be of degree 8, but the operator which transforms everything into its inverse must be of degree 12.

therefore one of them must be invariant under (ST). However, if G_{24} is written on four letters the operator (ST) is of degree four. Therefore such a group is impossible.

The only other possibility, if p_1 is 2, is that H is of order 2. Then $T_1T_2 = T_1T_3 = T_2T_3$ and T_1, T_2, and T_3 are the same. In this case, however, H *is identity.* Hence we have the

THEOREM: *A necessary and sufficient condition that there exist a group G generated by S and T satisfying the conditions.*

(1) $S^3 = T^2 = (ST)^8 = 1$,

(2) $(TS^2TS)^6 = 1$,

(3) *the commutator subgroup H of $\{S,(ST)^4\}$ is invariant under G,*

is that the order of H be $p_1^{n_1} \cdot p_2^{n_2} \cdot \cdots p_k^{n_k}$, where the n's are even numbers and the p's are of the form $8h+1$ or $8h+3$.

5. *H non-invariant.* We have seen that the group $\{S, T_1\}$ contains an abelian subgroup H. The groups which we have studied are those for which H is invariant under G. Let us suppose now that H is not invariant under G, and denote its conjugates by $H_1(=H)$, $H_2 \cdots$, H_n. If H contains a subgroup which is invariant under G this subgroup must satisfy the conditions determined for H in the preceding pages. Moreover, the quotient group of G with respect to this invariant subgroup contains n subgroups H'_i corresponding to the H's, which are not invariant and contain no invariant subgroup. Hence the quotient group can be represented on symbols for the n subgroups H_i. Therefore the group G of lowest order for which H is not invariant can be represented on symbols for the H's. Then the group $\bar{H}$ generated by the H's cannot be abelian, for it would leave each of the H's invariant. Then since (ST) does not leave H_1 fixed, and since T_1 does leave H_1 fixed but permutes the H's according to an operator of order 2, it follows that n must be at least 10. It is easy to prove by similar considerations that n must be at least 12. Hence

If H is not invariant under G, then $\bar{H}$ is not abelian, and H_1 is one of at least 12 conjugates.

Since practically nothing is known concerning non-abelian groups whose groups of isomorphisms contain subgroups isomorphic with G_{24} the consideration of H non-invariant promises to be a large undertaking and to involve considerations that have not entered in the preceding pages.

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ON A NEW NORMAL FORM OF THE GENERAL CUBIC SURFACE.

By ARNOLD EMCH.

1. *Introduction.* Mapping processes are not uncommonly used to give information on, or to facilitate the investigation of certain geometric problems. I refer for example to Segre's cubic variety in S_4 and its relation to the Kummer quartic surface, or to the Veronese surface based upon the system of conics in a plane with applications to Steiner's Roman surface, the theory of plane quartics, etc.

Such problems arise for example also in the mapping of projective spaces S_r upon rational hypersurfaces in S_{r+1}. Associating the variables (x) and (y) to these spaces respectively, the mapping may be accomplished by

$$\begin{aligned} &\rho y_1 = x_2 x_3 \cdots x_{r+1}, \quad \rho y_2 = x_1 x_3 \cdots x_{r+1}, \\ &\qquad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\rho y_{r+1} = x_1 x_2 \cdots x_r, \quad \rho y_{r+2} = x_1{}^r x_2{}^r + \cdots x^r{}_{r+1}. \end{aligned} \tag{1}$$

By eliminating the x's it is found that S_r is mapped upon the hypersurface S_r':

$$y_2{}^r y_3{}^r \cdots y_{r+1}{}^r + y_1{}^r y_3{}^r \cdots y_{r+1}{}^r + \cdots y_1{}^r y_2{}^r \cdots y_r{}^r - (y_1 y_2 \cdots y_{r+1})^{r-1} y_{r+2} = 0 \tag{2}$$

of order r^2 in $S_{r+1}(y)$. It is easily recognized that the mapping of S_r upon S_r' is birational, moreover that spaces with an even or odd number of dimensions are respectively mapped upon hypersurfaces of even and odd order. Restricting the problem to an Euclidean space by putting $y_{r+2} = 0$, that spaces have maps of finite or infinite extent according as they are of even or odd dimensions. This fact is related to the unilateralness or bilateralness of projective S_{2k} or S_{2k+1} respectively.

As a more detailed study of the transformation (1) will appear elsewhere, I shall restrict myself to some interesting applications.

One that is already well known is the mapping of a plane S_2 upon a Steiner surface S_2':

$$y_2{}^2 y_3{}^2 + y_3{}^2 y_1{}^2 + y_1{}^2 y_2{}^2 - y_1 y_2 y_3 y_4 = 0, \tag{3}$$

or conversely. To the plane sections $\Sigma a_i y_i = 0$, $(i = 1, 2, 3, 4)$ of S_2' corresponds in S_2 an ∞^3 system of conics

$$a_1 x_2 x_3 + a_2 x_3 x_1 + a_3 x_1 x_2 + a_4 (x_1{}^2 + x_2{}^2 + x_3{}^2) = 0. \tag{4}$$

To the ∞^2 sections by tangent-planes corresponds in S_2 a net of couples of lines among which are four double lines corresponding to the tropes of S_2'. The degenerate order conics of this net are apolar to the ∞^1 class-conics inscribed in the quadrilateral of these double lines. The latter are moreover the invariant lines in the quadratic line transformations $\rho u_1' = u_2 u_3$, $\rho u_2' = u_3 u_1$, $\rho u_3' = u_1 u_2$ in S_2. These few indications suffice to show the importance of this mapping process.

In this paper I shall consider the mapping of S_3 upon the nonic hypersurface S_3':

$$(5) \qquad y_2^3 y_3^3 y_4^3 + y_1^3 y_3^3 y_4^3 + y_1^3 y_2^3 y_4^3 + y_1^3 y_2^3 y_3^3 - y_1^2 y_2^2 y_3^2 y_4^2 y_5 = 0,$$

and its connection with the theory of cubic surfaces.

2. *The cubic surface as the map of a hyperplane in S_4.* It is obviously not difficult to establish corresponding figures in S_3 and S_3' by means of the transformation (1) when $r = 3$, i. e.,

$$(6) \qquad \begin{aligned} \rho y_1 = x_2 x_3 x_4, \quad \rho y_2 = x_1 x_3 x_4, \quad \rho y_3 = x_1 x_2 x_4, \quad \rho y_4 = x_1 x_2 x_3, \\ \rho y_5 = x_1^3 + x_2^3 + x_3^3 + x_4^3. \end{aligned}$$

Thus, to a line $l: \delta x_i = a_i + \lambda b_i$, $(i = 1, 2, 3, 4)$ in S_3 corresponds in S_3' the space-cubic

$$\tau y_i = K_0^{(i)} \lambda^3 + K_1^{(i)} \lambda^2 + K_2^{(i)} \lambda + K_3^{(i)}, \qquad (i = 1, \cdots, 5),$$

which lies on a certain hyperplane $\Sigma c_i y_i = 0$, and which is uniquely determined by the line l. A plane in S_3 may be represented parametrically by

$$\rho x_1 = \lambda_1 c_1, \ \rho x_2 = \lambda_2 c_2, \ \rho x_3 = \lambda_3 c_3, \ \rho x_4 = \lambda_1 + \lambda_2 + \lambda_3,$$

or explicitly by

$$(7) \qquad c_2 c_3 x_1 + c_3 c_1 x_2 + c_1 c_2 c_3 - c_1 c_2 c_3 x_4 = 0.$$

To it corresponds in parametric representation

$$(8) \qquad \begin{aligned} \delta y_1 = c_2 c_3 \lambda_2 \lambda_3 (\lambda_1 + \lambda_2 + \lambda_3), \quad \delta y_2 = c_3 c_1 \lambda_3 \lambda_1 (\lambda_1 + \lambda_2 + \lambda_3), \\ \delta y_3 = c_1 c_2 \lambda_1 \lambda_2 (\lambda_1 + \lambda_2 + \lambda_3), \quad \delta y_4 = c_1 c_2 c_3 \lambda_1 \lambda_2 \lambda_3, \\ \delta y_5 = c_1^3 \lambda_1^3 + c_2^3 \lambda_2^3 + c_3^3 \lambda_3^3 + (\lambda_1 + \lambda_2 + \lambda_3)^3, \end{aligned}$$

or also the intersection of the hypercone

$$(9) \qquad c_2 c_3 y_2 y_3 y_4 + c_3 c_1 y_1 y_3 y_4 + c_1 c_2 y_1 y_2 y_4 - c_1 c_2 c_3 y_1 y_2 y_3 = 0$$

with the hypernonic (5). This is a surface of order three since (9) and (5) intersect in a surface of order 27. But (5) has the six planes $(x_i = 0, x_k = 0)$,

$i \neq k$, $i, k = 1, 2, 3, 4$, as fourfold planes and (9) has these as single planes in S_4, so that the residual surface of intersection is indeed of order $27 - 4 \cdot 6 = 3$.

Conversely, to the intersection of the hyperplane

$$c_1y_1 + c_2y_2 + c_3y_3 + c_4y_4 + c_5y_5 = 0 \tag{10}$$

with (5) corresponds in S_3 the cubic surface F_3

$$c_1x_2x_3x_4 + c_2x_1x_3x_4 + c_3x_1x_2x_4 + c_4x_1x_2x_3 + c_5(x_1^3 + x_2^3 + x_3^3 + x_4^3) = 0, \tag{11}$$

depending evidently upon four effective constants.

Now the 19 effective constants of a general cubic surface may be reduced to 4 effective constants by collineation in S_3, which thus indicates, as far as counting of constants is concerned, that *the general cubic surface may be written in the form* (11). That this is true will be proved subsequently.

3. *The configuration Δ_{18} connected with the normal form* (11). The reduction to this normal form depends upon the properties of a configuration of 18 points Δ_{18} which is common to all cubics of the ∞^4 system (11). That this configuration is of a very special type appears from the fact that ordinarily 18 generic points determine a pencil of cubics, while here there are 5 linearly independent cubics on the Δ_{18}. The intersections of this system with the coördinate planes are syzygetic pencils. For example on $x_4 = 0$ we get

$$x_1^3 + x_2^3 + x_3^3 + (c_4/c_5)x_1x_2x_3 = 0, \tag{12}$$

with its fixed flexes on the three sides A_2A_3, A_3A_1, A_1A_2, and the three real flexes on $x_1 + x_2 + x_3 = 0$. The same situation exists on the three remaining coördinate planes. The flexes on A_1A_2 for example are obtained by putting $x_3 = 0$, $x_4 = 0$, so that $x_1^3 + x_2^3 = 0$ determines on A_1A_2 the flexes

$$(1, -1, 0, 0); \quad (1, -w, 0, 0); \quad (1, -w^2, 0, 0); \quad w^3 = 1.$$

In a similar manner are found three flexes on each of the remaining 5 edges A_iA_k. We thus obtain a configuration Δ_{18} of 18 such flexes through which all cubics of the system (11) pass. This configuration is invariant under the Abelian collineation group G_{27}.

$$x_1' = x_1, \quad x_2' = w^\alpha x_2, \quad x_3' = w^\beta x_3, \quad x_4' = w^\gamma x_4,$$

$\alpha, \beta, \gamma = 0, 1, 2$. But as together the 4 syzygetic pencils are invariant under the symmetric group G_{24} on x_1, x_2, x_3, x_4, *the entire configuration Δ_{18} is invariant under a collineation group $G_{27 \cdot 24} = G_{648}$ of order* 648.

To establish the properties of Δ_{18}, consider the plane $x_1 + x_2 + x_3$

$+ x_4 = 0$ which contains the four lines cut out by $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$. These intersect in 6 points $(1, -1, 0, 0)$, . . . which are flexes of the configuration of 18 points. Each of these lines, for example $x_1 + x_2 + x_3 + x_4 = 0$, $x_2 + x_3 + x_4 = 0$ determines a pencil of planes

$$x_1 + (\lambda + 1)x_2 + (\lambda + 1)x_3 + (\lambda + 1)x_4 = 0.$$

For $\lambda = 0$ we get the unit plane. For $\lambda + 1 = w$ we have,

$$x_1 + wx_2 + wx_3 + wx_4 = 0,$$

in which lie the six points $(0, 1, -1, 0)$; $(0, 0, 1, -1)$; $(0, 1, 0, -1)$; $(1, -w^2, 0, 0)$; $(1, 0, -w^2, 0)$; $(1, 0, 0, -w^2)$, which belong to the Δ_{18} and which are the six vertices of a quadrilateral of Maclaurin lines (3 collinear flexes). For $1 + \lambda = w^2$ we obtain the plane $x_1 + w^2x_2 + w^2x_3 + w^2x_4 = 0$ in which lies likewise such a quadrilateral. Thus through every Maclaurin line not identical with an edge A_iA_k there are three planes each containing a quadrilateral formed by Maclaurin lines from the Δ_{18}. There are $(12 - 3)4 = 36$ such lines, hence 3×36 such planes. But as each of these planes is on 4 Maclaurin lines, there are altogether $3 \times 36 : 4 = 27$ such planes. These are obtained from $\Sigma x_i = 0$ by the substitutions of the G_{27}. The result may be stated in the

THEOREM. *The* 18 *points of the configuration* Δ_{18} *lie by* 6 *vertices of a quadrilateral in* 27 *planes, not including the coördinate planes. Through each Maclaurin line different from an* A_iA_k *there are* 3 *planes each containing such a quadrilateral. The entire* Δ_{18} *is invariant under a collineation group of order* 648.

That there is a finite number of Δ_{18} configurations on the general cubic surface will be proved in the next section.

4. *Reduction of the general cubic surface to the* Δ_{18} *normal form.* It is necessary to first point out an important property of the configuration $\Delta_{9,12}$ of the nine flexes of a general cubic and its twelve Maclaurin lines. By collineation such a cubic may always be reduced to the Hessian normal form

$$x_1^3 + x_2^3 + x_3^3 - 3\lambda x_1x_2x_3 = 0. \tag{13}$$

The $\Delta_{9,12}$ is independent of the parameter λ, from which follows that these configurations for all general cubics are projectively equivalent. This can also be proved directly as follows:

The real flex-tangents of (13) at $(1, -1, 0)$; $(1, 0, -1)$; $(0, 1, -1)$ are

$$\begin{aligned} \rho x_1' &= \lambda x_1 + x_2 + x_3 = 0, \\ \rho x_2' &= x_1 + \lambda x_2 x_3 = 0, \\ \rho x_3' &= x_1 + x_2 + \lambda x_3 = 0. \end{aligned} \tag{14}$$

If these are chosen as the sides of a new coördinate triangle, (13) assumes the form

$$(x_1' + x_2' + x_3')^3 - 3\frac{(\lambda+2)^3}{\lambda^2+\lambda+1}(x_1' + x_2' + x_3') = 0. \tag{15}$$

(14) may be interpreted as a perspective with (1, 1, 1) as a center and $x_1 + x_2 + x_3 = 0$ as the pointwise invariant axis. Now choose arbitrarily three lines

$$\begin{aligned} \rho x_1' &= \lambda_1(x_1 + x_2 + x_3) + \mu x_1 = 0, \\ \rho x_2' &= \lambda_2(x_1 + x_2 + x_3) + \mu x_2 = 0, \\ \rho x_3' &= \lambda_3(x_1 + x_2 + x_3) + \mu x_3 = 0, \end{aligned} \tag{16}$$

as flex-tangents of a new cubic at the points (1, —1); (1, 0, —1); (0, 1, —1) respectively, which will be of the form

$$\begin{aligned} (x_1 + x_2 + x_3)^3 - 3\gamma[\lambda_1(x_1 + x_2 + x_3) + \mu x_1] \\ \times [\lambda_2(x_1 + x_2 + x_3) + \mu x_2]\ [\lambda_3(x_1 + x_2 + x_3) + \mu x_3] = 0. \end{aligned} \tag{17}$$

(16) may again be interpreted as a perspective with $x_1 + x_2 + x_3 = 0$ as an axis and $(\lambda_1, \lambda_2, \lambda_3)$ as a center. From (16)

$$\rho(x_1' + x_2' + x_3') = (x_1 + x_2 + x_3)(\lambda_1 + \lambda_2 + \lambda_3 + \mu),$$

so that by (16) (17) is transformed into

$$(x_1' + x_2' + x_3')^3 - 3\gamma(\lambda_1 + \lambda_2 + \lambda_3 + \mu)^3 x_1' x_2' x_3', \tag{18}$$

or into one of the cubics of the pencil (15). The three real flexes of (15) are identical with those of (13). Moreover two of the vertices of all flex-triangles of (15): $(1, w, w^2)$ and $(1, w^2, w)$ coincide with those of (13). Hence the

THEOREM. *All cubics of the pencil*

$$x_1 + x_2 + x_3)^3 - 3mx_1x_2x_3 = 0$$

have perspective flex-configuration $\Delta_{9,12}$. *Moreover two cubics with real flexes at* (0, 1, —1); (1, 0, —1); (1, —1, 0) *have perspective flex-configurations with* $x_1 + x_2 + x_3 = 0$ *as the axis of perspective.*

This proposition can evidently be extended to the case where the axis of perspective is any of the 12 Maclaurin lines.

Now assume two plane cubics $C_3^{(1)}$ and $C_3^{(2)}$ in S_3 with the two real flex-triangles Δ_1 and Δ_2 respectively, situated in two planes p_1 and p_2 which intersect in a line l. Let Δ_1 and Δ_2 have l as a common side and on it three common flex-points W_1, W_2, W_3 (one real, two imaginary), so that the flex-tangents of $C_3^{(1)}$ and $C_3^{(2)}$ at these points determine the entire flex-configurations $\Delta_{9,12}$ for both curves. According to the foregoing theorem these are perspective with l as the axis of perspective. Choosing Δ_1 as $A_1A_2A_3$ and Δ_2 as $A_1A_2A_4$ of a coördinate tetrahedron, then F_3 has the form

$$x_1^3x_2^3x_3^3 - 3a_4x_1x_2x_3 - 3x_4(a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{44}x_4^2 + a_{14}x_1x_4 + a_3x_1x_2 + a_1x_2x_3 + a_2x_1x_3 + a_{24}x_2x_4 + a_{34}x_3x_4) = 0.$$

For $x_3 = 0$ this must reduce to $x_1^3 + x_2^3 + x_4^3 - 3a_3x_1x_2x_4 = 0$, which is only true when $a_{11} = 0$, $a_{22} = 0$, $a_{33} = 0$, $a_{44} = 1/3$, $a_{14} = 0$, $a_{24} = 0$, $a_{34} = 0$. Thus F_3 has the form

$$(19) \quad x_1^3 + x_2^3 + x_3^3 + x_4^4 - 3(a_1x_2x_3x_4 + a_2x_1x_3x_4 + a_3x_1x_2x_4 + a_4x_1x_2x_3) = 0.$$

The existence of two plane cubics $C_3^{(1)}$ and $C_3^{(2)}$, as described above, on a cubic surface thus necessarily leads to the form (19).

We next determine the manifold of cubic surfaces F_3 which may be constructed in this fashion. There are ∞^1 planes p_1 on a line l in S_3. The line l may be chosen as one side of a real flex-triangle Δ_1. The two other sides of Δ_1 and the line of the 3 real flexes may be chosen in ∞^6 ways. In another plane p_2 on l we can choose a perspective flex-configuration in ∞^3 ways. The planes p_1 and p_2 or l can be chosen in ∞^2 ways, hence the two perspective flex-configuration $(\Delta_1)(\Delta_2)$ in $\infty^{6+3+2} = \infty^{11}$ ways. On each of these there are ∞^4 Δ_{18} cubic surfaces (19), hence on l ∞^{15} such surfaces. As l may be chosen in ∞^4 ways, the manifold of Δ_{18} cubics is therefore ∞^{19}, i. e., identical with that of cubics F_3 in S_3. This may be verified as follows: There are $\infty^{3.5} = \infty^{15}$ pentahedrons (tetrahedron + unit point), hence $\infty^{15} \cdot \infty^4 = \infty^{19}$ Δ_{18} cubics F_3.

The unit-plane cuts F_3 in a cubic on which lies a hexagon formed by the vertices of a quadrilateral cut out by the coördinate planes. On each side of this quadrilateral lie three flexes such that the flex-tangents lie in the corresponding coördinate plane. The existence of this figure again insures the form (19) of the cubic surface. A generic plane p cuts the cubic surface in a plane cubic C_3 in which may be inscribed ∞^1 quadrilateral hexagons (A_{ik}), where A_{ik} is the intersection of two sides of a quadrilateral l_1, l_2, l_3, l_4. Consider the asymptotic tangents $t_{i1}^{(1)}$, $t_{i1}^{(2)}$; $t_{i2}^{(1)}$, $t_{i2}^{(2)}$; $t_{i3}^{(1)}$, $t_{i3}^{(2)}$ to the

surface at the points A_{i1}, A_{i2}, A_{i3} on the line l_i. Through l_i and $t_{i1}^{(1)}$ pass a plane h, then the condition that $t_{i2}^{(1)}$ shall lie on h absorbs one constant. As $t_{i1}^{(1)}$ and $t_{i2}^{(1)}$ are two flex-tangents of the plane cubic cut out by h, one of the third pair, say $t_{i3}^{(1)}$, will necessarily become a flex-tangent of this cubic. Thus the condition that $t_{i1}^{(1)}$, $t_{i2}^{(1)}$, $t_{i3}^{(1)}$ shall lie in a plane requires one constant. The same argument may be repeated for each of the three remaining lines of the quadrilateral. Thus a quadrilateral (like in the unit-plane of the Δ_{18} cubic surface) requires four conditions and as there are ∞^4 such plane inscribed quadrilaterals on a cubic surface, there can exist only a finite number of Δ_{18} configurations on a general cubic surface. The result may be stated in the

THEOREM: *The manifold of cubic surfaces on all Δ_{18} configurations of S_3 is identical with the manifold of general cubic surfaces in S_3. From this follows that every general cubic surface may be represented by an equation in which the terms $x_i^2 x_k$ are missing.*

In this representation the cubic monoids are included as will be seen in the next section.

5. *The 27 lines on the cubic surface.* The tetrahedron on Δ_{18} cuts F_3 in four plane cubics C_1, C_2, C_3, C_4, so that each of the 27 lines on F_3 is a quadrisecant of this system of four cubics. In the theory of ruled surfaces the following theorem is known: Let C_1, C_2, C_3 be three curves of order m_1, m_2, m_3 respectively, and let C_1 and C_2 have s_3, C_2 and C_3 have s_1, C_3 and C_1 have s_2 points in common, then the trisecants which each cut C_1, C_2, C_3 once, generate a ruled surface R of order $n = 2m_1m_2m_3 - (s_1m_1 + s_2m_2 + s_3m_3)$ with C_1, C_2, C_3 as $(m_2m_3 - s_1)$—, $(m_3m_1 - s_2)$—, $(m_1m_2 - s_3)$— fold curves respectively. If we apply this to the three plane cubics C_1, C_2, C_3 on Δ_{18}, we obtain a ruled surface R of order $2\cdot3\cdot3\cdot3 - (3\cdot3 + 3\cdot3 + 3\cdot3) = 27$ with each C_1, C_2, C_3 as a 6-fold curve. C_1, C_2, C_3 cut C_4 in 9 points which in counting the proper intersections of R with C_4 must be subtracted as sixfold points. Hence R cuts C_4 in $3.27 - 6.9 = 27$ points outside of the flexes of C_4. From this follows *that the four curves $C_1C_2C_3C_4$ admit of 27 quadrisecants which therefore lie on F_3. This again proves the generic cubic surface on Δ_{18} as a general cubic.*

The foregoing procedure may be applied in precisely the same manner to the four plane cubics cut out on a general F_3 by the faces of a generic tetrahedron and constitutes thus a novel prove for the existence of 27 lines on a cubic.

6. *Cubic Monoids.* To a generic point $P(a_1, a_2, a_3, a_4)$ in S_3 corre-

sponds on S_3' by (6) a point $P'(a_2a_3a_4,\ a_1a_3a_4, \cdots)$. The tangent hyperplane at this point is easily found as

$$(20)\qquad \begin{aligned} &a_1(a_2^3+a_3^3+a_4^3-2a_1^3)y_1+a_2(a_1^3+a_3^3+a_4^3-2a_2^3)y_2\\ &+a_3(a_1^3+a_2^3+a_4^3-2a_3^3)y_3\\ &+a_4(a_1^3+a_2^3+a_3^3-2a_4^3)y_4-a_1a_2a_3a_4y_5=0. \end{aligned}$$

To this corresponds inversely in S_3 the cubic surface

$$(21)\quad \begin{aligned} M_3=\ &a_1(a_2^3+a_3^3+a_4^3-2a_1^3)x_2x_3x_4+a_2(a_1^3+a_3^3+a_4^3-2a_2^3)x_1x_3x_4\\ &+a_3(a_1^3+a_2^3+a_4^3-2a_3^3)x_1x_2x_4\\ &+a_4(a_1^3+a_2^3+a_3^3-2a_4^3)x_1x_2x_3-a_1a_2a_3a_4(x_1^3+x_2^3+x_3^3+x_4^3)=0, \end{aligned}$$

which has a double point at $P(a_1, a_2, a_3, a_4)$ and is therefore a cubic monoid. M_3 depends upon three effective constants. Every point (a) in S_3 determines such a monoid uniquely and is the double point of M_3. We have thus the

THEOREM: *The generic points of S_3 as double points determine the ∞^3 cubic monoids on the configuration Δ_{18}* uniquely.

If in M_3 we leave (x) fixed and let (a) vary we have obviously the

THEOREM: *The locus of the double points of all cubic monoids on Δ_{18} through a fixed generic point is a quartic surface.*

To the intersection of a hyperplane $\Sigma c_iy_i=0$ with S_3' corresponds in S_3 the cubic surface F_3 in a normal form on Δ_{18}. Among these are evidently the monoids M_3. The discriminant D of F_3 is a homogeneous polynomial of degree 32 in the c's, which puts a condition on the c's. From (20) it is evident that this discriminant may be expressed parametrically by

$$(22)\qquad \begin{aligned} \rho c_1&=a_1(a_2^3+a_3^3+a_4^3-2a_1^3)\\ \rho c_2&=a_2(a_1^3+a_3^3+a_4^3-2a_2^3)\\ \rho c_3&=a_3(a_1^3+a_2^3+a_4^3-2a_3^3)\\ \rho c_4&=a_4(a_1^3+a_2^3+a_3^3-2a_4^3)\\ \rho c_5&=-a_1a_2a_3a_4 \end{aligned}$$

as a rational hypersurface of order 32 in a projective hyperspace $S_4(c_1, c_2, c_3, c_4, c_5)$.

This may be verified by making use of the pentahedral normal form

$$(23)\qquad \Sigma c_ix_i^3=0,\ \Sigma x_i=0,\ i=1, 2, 3, 4, 5.$$

with the discriminant *

* Salmon-Fiedler, *Analytische Geometrie des Raumes*, Vol. 2 (1874), pp. 367-368.

$$\pm\sqrt{c_2c_3c_4c_5}\pm\sqrt{c_1c_3c_4c_5}\pm\cdots\pm\sqrt{c_1c_2c_3c_4}=0, \tag{24}$$

which may be expressed parametrically by

$$\begin{aligned}
\rho c_1 &= \lambda_2^2\lambda_3^2(\lambda_1+\lambda_2+\lambda_3+1)^2,\\
\rho c_2 &= \lambda_3^2\lambda_1^2(\lambda_1+\lambda_2+\lambda_3+1)^2,\\
\rho c_3 &= \lambda_1^2\lambda_2^2(\lambda_1+\lambda_2+\lambda_3+1)^2,\\
\rho c_4 &= \lambda_1^2\lambda_2^2\lambda_3^2(\lambda_1+\lambda_2+\lambda_3+1)^2,\\
\rho c_5 &= \lambda_1^2\lambda_2^2\lambda_3^2,
\end{aligned}$$

These results may be stated in the

THEOREM: *The discriminant of the cubic surface in the* Δ_{18} —, *or also the pentahedral normal form, may be represented by a rational hypersurface of order* 32 *in a projective* S_4.

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ON PLANE CREMONA TRIADIC CHARACTERISTICS.

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The problem of finding plane Cremona transformations having only three groups of basis points, that is, transformations with triadic characteristics, has been attacked by Ruffini,[1] Montesano,[2] and Farnum.[3] In this paper twelve infinite sequences of two-parameter triadics are obtained, all of which have for one of their groups a single point of highest multiplicity. These triadics result directly from the development of a new procedure [4,5,6] whereby all geometric characteristics may be found. The principal feature of this procedure is its perfect regularity and simple explicit formulation.

1. It is a well-known theorem that if P, Q, and R are basis points of multiplicities $p \geqq 0$, $q \geqq 0$, and $r \geqq 0$ of a plane Cremona transformation T_n of order n and also are the basis points of a quadratic inversion T_2 with which T_n is compounded, the resultant transformation $T_{n'}$ of order n' has the same number of basis points at T_n of the same multiplicities except that it loses the points P, Q, and R, and gains three basis points P', Q', and R' of multiplicities $p' = p + e$, $q' = q + e$, and $r' = r + e$, while $n' = n + e$, where $e = n - (p + q + r)$. It will be convenient to regard the three points gained by $T_{n'}$ as the transforms of the three points lost by T_n.

2. We give the symbol P a particular connotation. If a transformation is to be successively compounded with T_2's and one of its basis points is called P, it is intended that P and each of its transforms are to be successively used as common basis points. If T_n is compounded with a T_2 having P, Q_1, and

1 "Sulla risoluzione delle trasformazioni Cremoniane," *Memorie Istituto Bologna*, Series 3, Vol. 8 (1877), pp. 457-525.

2 "I gruppi Cremoniani di numeri," *Atti Accademia Napoli*, Series 2, Vol. 15 (1914), No. 7.

3 "On triadic Cremona nets of plane curves," *American Journal of Mathematics*, Vol. 50 (1928), p. 357.

4 Montesano, "Su le rete omaloidiche di curve," *Rendiconti Accademia Napoli*, Series 3, Vol. 11 (1905), pp. 259-303.

5 Montesano, "Su i quadri caratteristici delle corrispondenze birazionali piane," *Rendiconti Accademia Napoli*, Series 3, Vol. 21 (1915), pp. 30-38, 69-79, 113-119.

6 Montesano, "Su alcuni problemi fondamentali nella teoria delle corrispondenze Cremoniane," *Rendiconti Accademia Napoli*, Series 3, Vol. 34 (1928), pp. 42-50, 89-97, 123-136.

R_1 as common basis points, $e = (n-p) - (q_1 + r_1)$, $n' = n + e$, $p' = p + e$, and $n' - p' = n - p$. If the resultant transformation $T_{n'}$ is then compounded with a T_2 having P', Q_2, and R_2 as common basis points, $e' = (n' - p') - (q_2 + r_2) = (n - p) - (q_2 + r_2)$. A sufficient condition that $e' = e$ is that $q_2 = q_1$ and $r_2 = r_1$. Hence, in a succession of compoundings involving a point P and its transforms, $(n - p)$ is invariant, and e is invariant if Q and R are each repeatedly taken with invariant multiplicity.

3. The characteristic (1) is known to be geometric. Let P be the single point of its first group. Compound it with a T_2 having its other two basis points general [i. e., of multiplicity zero in (1)]. The resultant is (2) since $e = x_{01}$. If this compounding is repeated, (2) is altered in the same way as was (1). The resultant of x_{11} of these compoundings is (3). In general, compounding with T_2's each having one basis point on P and two general will be referred to as compounding in the first way.

(1)	$n=[x_{01}+1]$;	1^1,	$1^{x_{01}}$,	$(2x_{01}-1)^1$.
(2)	$n=[x_{01}+1+x_{01}]$;	$1^{1+x_{01}}$,	$(1+2)^{x_{01}}$,	$(2x_{01}-1)^1$.
(3)	$n=[x_{01}+1+x_{01}x_{11}]$;	$1^{1+x_{01}x_{11}}$,	$(1+2x_{11})^{x_{01}}$,	$(2x_{01}-1)^1$.
(4)	$n=[x_{01}+1+x_{01}x_{11}-1]$;	$1^{1+x_{01}x_{11}-1}$,	$(1+2x_{11}-1)^{x_{01}}$, $1^{x_{01}-1}$,	$(2x_{01}-1-1)^1$.
(5)	$n=[x_{01}+1+x_{01}x_{11}-x_{12}]$;	$1^{1+x_{01}x_{11}-x_{12}}$,	$(1+2x_{11}-x_{12})^{x_{01}}$, $x_{12}{}^{x_{01}-1}$,	$(2x_{01}-1-x_{12})^1$.

4. Compound (3) with a T_2 having one basis point on the last transform of P, one on a point of the second group, and one on a point of the third. The resultant is (4) since $e = -1$. If this compounding is repeated, (4) is altered in the same way as was (3). The resultant of x_{12} of these compoundings is (5). In general, compounding with T_2's each having one basis point on P, a second on a point of the second group, and the third on a point of any other group will be referred to as compounding in the second way.

5. The procedure for obtaining all geometric characteristics may be described in general terms as follows: suppose the characteristic

$$(6) \qquad n;\ 1^{z_{k1}},\ (1+y_{k2})^{z_{k2}},\ y_{k3}^{z_{k3}}, \cdots, y_{ki}^{z_{ki}}, \cdots, y_{k2^{k+1}}^{z_{k2^{k+1}}}$$

to be geometric, with $n = z_{k1} + z_{k2}$, where the y's and z's are polynomials in x_{rs}, $r = 0, 1, \cdots, k$, and $s = 1, 2, \cdots, 2^r$. An illustration of such a characteristic is (5). Take for P a point of multiplicity z_{k2} (involving a change of P) and write (6) in the form (7) so that P constitutes the first group.

$$(7) \qquad n;\ 1^{z_{k2}},\ 1^{z_{k1}},\ y_{k2}^{z_{k2}},\ y_{k3}^{z_{k3}}, \cdots, y_{ki}^{z_{ki}}, \cdots, y_{k2^{k+1}}^{z_{k2^{k+1}}}.$$

$$(8)\qquad n' = [n + z_{k1}x_{k+1,1}];\quad 1^{z_{k2}+z_{k1}x_{k+1,1}},\quad (1 + 2x_{k+1,1})^{z_{k1}}.$$
$$y_{k2}^{z_{k2}}, \cdots, y_{ki}^{z_{ki}}, \cdots, y_{k2^{k+1}}^{z_{k2^{k+1}}}$$

$$(9)\qquad n' = [n + z_{k1}x_{k+1,1} - \sum_{i=2}^{2^{k+1}} z_{ki}x_{k+1,i}];\quad 1^{z_{k2}+z_{k1}x_{k+1,1} - \sum_{i=2}^{2^{k+1}} z_{ki}x_{k+1,i}},$$
$$(1 + 2x_{k+1,1} - \sum_{i=2}^{2^{k+1}} x_{k+1,i})^{z_{k1}},\ x_{k+1,2^{k+1}}^{z_{k1}-z_{k2}},\cdots,\ x_{k+1,i}^{z_{k1}-z_{ki}},\cdots,$$
$$x_{k+1,2}^{z_{k1}-z_{k2}},\ (y_{k2} - x_{k+1,2})^{z_{k2}},\cdots,\ (y_{ki} - x_{k+1,i})^{z_{ki}},\cdots,$$
$$(y_{k2^{k+1}} - x_{k+1,2^{k+1}})^{z_{k2^{k+1}}}.$$

Compound (7) in the first way $x_{k+1,1}$ times, where $e = z_{k1}$, getting (8). Compound (8) in the second way $x_{k+1,i}$ times with the third basis point of the compounding T_2 on a point of multiplicity z_{ki}, i having successively all integer values from 2 to 2^{k+1}, getting (9), since $e = -z_{ki}$. As (9) is of the same form as (6) the procedure is inductive. The set of compoundings used to obtain (9) from (6) will be referred to as one step of the inductive procedure, and (6) itself will be referred to as the general characteristic resulting from k steps of this procedure. It should be noted that each general characteristic includes all preceding ones.

The only condition on the x's is that all the base numbers (i. e., the expressions giving the number of points in each group) must be non-negative, and if this condition is satisfied all characteristics obtained by this procedure are geometric, since (1) is geometric.

6. THEOREM. *Every plane Cremona geometric characteristic can be obtained as the result of sufficiently many steps of the above procedure* [*starting from* (1)] *and proper evaluation of the x's.*

This theorem will be proved by showing that the inverse of the procedure will reduce a given characteristic to a particular case of (1). Let the characteristic of a given transformation T be n; $y_h{}^h$, $y_{h-1}^{h-1}, \cdots, y_1{}^1$, where h denotes the highest multiplicity in T. (In the course of this proof h will also denote the highest multiplicity in any particular transform of T.) Take one of the points of multiplicity h as P, and let the multiplicity of this particular point be denoted by p. The reason for the notation is that $(n - p)$ is always invariant, but $(n - h)$ is not.

7. If there exists in T a point Q (other than P) of multiplicity q, $n - p > q > (n - p)/2$, compound T with a T_2 having one basis point on P, one on Q, and one general. In the notation of paragraph 1, $e > 0$, $p' \geqq q' = n - p$, and $r' < (n - p)/2$, so that this compounding is in the

inverse of the second way. For simplicity we drop the primes in considering this and succeeding transforms of T. In this way all points Q may be removed, so that there remain only P of multiplicity $p = h$, points of multiplicity $(n - p)$, and points of multiplicity not greater than $(n - p)/2$. Before making further compoundings we use

8. Noether's Extended Theorem. In any geometric characteristic the sum of the first, third, and fifth highest multiplicities is greater than the order. It follows that there are at least three points of multiplicity greater than $(n - h)/2$. Hence, if $p = h$ and there is no point Q, there are at least two points (other than P in the event that $p = n - p$) of multiplicity $(n - p)$, and $p \geqq n - p$.

9. Compound T in its present form with a T_2 having one basis point on P and two on points A and B of multiplicity $(n - p)$. In obvious notation $e = -(n - p)$ and $a' = b' = 0$, so that this compounding is in the inverse of the first way. Let this compounding be repeated as many times as possible, that is, until $p < n - p$.

If now y_{n-p} is positive, $n - p = h > p = n - h$, and the group consists of but a single point. Since $h + p = n$, p is the second highest multiplicity, and T has been reduced to a transformation T' of the form (6) with P of the proper multiplicity, so that, if the procedure produces T' it also produces T.

10. If, however, y_{n-p} is zero, then $p < h \leqq (n - p)/2$, since T now contains no point Q. Compound T in its present form with a T_2 having its basis points on P, a point H of multiplicity h, and a general point K. Since $h' = n - p > p' \geqq k' = e \geqq (n - p)/2$ this compounding is in the inverse of the second way. But P is now a point of second highest multiplicity and $y_{n-p} = 1$. Hence in this case also T has been reduced to a transformation T' of the form (6).

11. As this last compounding is the inverse of the second way its use here is unwarranted. But the difficulty may be overcome without affecting T'. Because of the invariance of $(n - p)$ the effect on T of each compounding is independent of the others, so that the order of the compoundings may be changed without altering T' and a compounding may be omitted without impairing the effect on T of any of the others. Let K' be the point of multiplicity k' involved in the last compounding. If K' was present in the original T as a point Q, the continued product of the compounding which removed it, the last compounding of paragraph 9, and the compounding of

paragraph 10, is the identity. Hence they may all be omitted without affecting T'. If K' was not present as a point Q, the compounding introducing it is necessary, but may be performed in paragraph 7.

12. Now take the single point of multiplicity $(n-p)$ as a new P and repeat the above process, noting that the new value of $(n-p)$ is less than the old one. By sufficiently many repetitions of this process it is possible to reduce the value of $(n-p)$ to one, and when this has been done the result must necessarily be a particular case of (1).[7]

13. This proof was conducted in such a way as to show that any characteristic may be obtained with P a point of highest multiplicity, not only in it, but throughout the process of getting it. Hence, in constructing T from (1) by the inverse of the above reduction, all x_{r1}, $r = 0, 1, 2, \cdots, k$ (where k is the required number of steps in the inductive procedure) are positive. The vanishing of an x_{r1} by no means invalidates the procedure; it leads only to duplicates. We shall assume in the sequel that all such x's are positive.

14. If we now impose the condition that all but three of the base numbers vanish in the general characteristic resulting from k steps of the inductive procedure, $k > 1$, it results that all the x's can be evaluated by six different sets of formulas in terms of x_{01} and x_{11} taken as independent parameters and in no other way. If, for successive values of k, these values of the x's are substituted in the general characteristics, triadics 1, 2, 3, 4, 5, and 6 result. (In these triadics we have written merely x_0 for x_{01} and x_1 for x_{11}.) We omit the proofs of these statements because of typographical difficulties.

Other formulas for evaluating the x's so that triadics result may be obtained by making two or more of the multiplicities equal in the general characteristics. For example, the condition $x_{01} = 2$ makes every other pair of multiplicities equal, so that four, or even five base numbers may be positive without involving more than three different multiplicities. However, this particular condition leads only to special cases of the triadics given below, together with some half-dozen isolated triadics.

15. Let $\cdots, u_{-2}, u_{-1}, u_0, u_1, u_2, \cdots$ be the two-way sequence of polynomials satisfying the relations

$$(A_e) \qquad u_{i+1} + u_{i-1} = x_1 u_i, \ i \text{ even},$$
$$(A_o) \qquad u_{i+1} + u_{i-1} = u_i, \ i \text{ odd},$$

and the condition that

$$u_0 = 0, \quad u_1 = 1.$$

7 Hudson, *Cremona Transformations*, p. 98.

It follows immediately that all the u's are polynomials in the one variable x_1 and that $u_i = -u_{-i}$, since $u_{-1} = -1$. These polynomials are connected by many relations, among which the following are used in the sequel:

(B_e) $\quad u_{i\pm2} \mp x_1 u_i - 2u_{i\pm1} = u_{i\mp2}$, i even, signs dependent.

(B_o) $\quad u_{i\pm2} + x_1 u_i - 2x_1 u_{i\pm1} = u_{i\mp2}$, i odd, signs dependent.

$(C1)$ $\quad u_{i+(j+1)}u_{i-(j+1)} = x_1 u_{i+j}u_{i-j} - u_{2j+1}$, $(i \pm j)$ even.

$(C2)$ $\quad x_1 u_{i+(j+1)}u_{i-(j+1)} = u_{i+j}u_{i-j} - u_{2j+1}$, $(i \pm j)$ odd.

$(C3)$ $\quad u_{i+(j+1)}u_{i-(j+1)-1} = u_{i+j}u_{i-j-1} - u_{2j+2}$, all $(i \pm j)$.

$(C4)$ $\quad u_{i+(j+2)}u_{i-(j+2)} = u_{i+j}u_{i-j} - u_{2j+2}$, $(i \pm j)$ even.

$(C5)$ $\quad u_{i+(j+2)}u_{i-(j+2)} = u_{i+j}u_{i-j} - x_1 u_{2j+2}$, $(i \pm j)$ odd.

$(C6)$ $\quad u_{i+(j+2)}u_{i-(j+2)-1} = u_{i+j}u_{i-j-1} - u_{2j+3}$, all $(i \pm j)$.

Relations (B) are proven directly by substitution from relations (A). Relation $(C1)$ is proven by induction first for the case $j = 0$. Then relations $(C2)$ and $(C3)$ are proven directly for the case $j = 0$ by substitution from relations (A) and $(C1)$. Finally relations $(C1)$, $(C2)$, and $(C3)$ are proven for general j by showing that if they all hold for all proper i when $j = s$, then they all hold for all proper i when $j = s + 1$. Relations $(C4)$, $(C5)$, and $(C6)$ are proven directly by substitution from relations (A) and the other relations (C). Details of the proofs are omitted because they offer no difficulty.

As a digression we may point out an interesting property of these u-polynomials. Transform u_i into u'_i by the substitution $x_1 = x'_1 + 2$. The roots of the equation $u'_i = 0$, where i is positive and even, are equal to $2\cos(2n\pi)/i$, $n = 1, 2, \cdots, (i-2)/2$.

16. The following triadics can be shown to be geometric and their conjugates can be determined by Montesano's method [6]:

(1)	$\begin{cases} n=[x_1b_ka_k+x_1+1]; \\ n=[\ \ b_ka_k+x_1+1]; \end{cases}$	$1^{(n-1)-x_1u_ka_k},$ $1^{(n-1)-\ u_ka_k}$	$(2x_0-1)^{x_1u_ka_k},$ $(2x_0-1)^{\ u_ka_k},$	$(2x_1)^{u_{k+1}a_k+1},$ $(2x_1)^{u_{k+1}a_k+1},$	k even. k odd.
(1c)	$\begin{cases} n=[x_1a_kb_k+x_1+1]; \\ n=[\ \ a_kb_k+x_1+1]; \end{cases}$	$1^{(n-1)-x_1u_kb_k},$ $1^{(n-1)-\ u_kb_k},$	$(2x_0-1)^{x_1u_kb_k},$ $(2x_0-1)^{\ u_kb_k},$	$(2x_1)^{u_{k-1}b_k+1},$ $(2x_1)^{u_{k-1}b_k+1},$	k even. k odd.
(2)	$\begin{cases} n=[x_1a_k^2+1]; \\ n=[\ \ a_k^2+1]; \end{cases}$	$1^{(n-1)-x_1u_ka_k},$ $1^{(n-1)-\ u_ka_k},$	$(2x_0-1)^{x_1u_ka_k},$ $(2x_0-1)^{\ u_ka_k},$	$(2x_1)^{u_{k-1}a_k},$ $(2x_1)^{u_{k-1}a_k},$	k even. k odd.
(2′)	$\begin{cases} n=[x_1b_k^2+1]; \\ n=[\ \ b_k^2+1]; \end{cases}$	$1^{(n-1)-x_1u_kb_k},$ $1^{(n-1)-\ u_kb_k},$	$(2x_0-1)^{x_1u_kb_k},$ $(2x_0-1)^{\ u_kb_k},$	$(2x_1)^{u_{k+1}b_k},$ $(2x_1)^{u_{k+1}b_k},$	k even. k odd.
(3)	$\begin{cases} n=[x_1c_k^2+1]; \\ n=[\ \ c_k^2+1]; \end{cases}$	$1^{(n-1)-x_1u_kc_k},$ $1^{(n-1)-\ u_kc_k},$	$(2x_0-1-2x_1)^{x_1u_kc_k},$ $(2x_0-1-2x_1)^{\ u_kc_k},$	$(2x_1)^{u_{k+1}c_k},$ $(2x_1)^{u_{k+1}c_k},$	k even. k odd.
(3′)	$\begin{cases} n=[x_1d_k^2+1]; \\ n=[\ \ d_k^2+1]; \end{cases}$	$1^{(n-1)-x_1u_kd_k},$ $1^{(n-1)-\ u_kd_k},$	$(2x_0-1-2x_1)^{x_1u_kd_k},$ $(2x_0-1-2x_1)^{\ u_kd_k},$	$(2x_1)^{u_{k-1}d_k},$ $(2x_1)^{u_{k-1}d_k},$	k even k odd.
(4)	$\begin{cases} n=[x_1d_kc_k+x_1+1]; \\ n=[\ \ d_kc_k+x_1+1]; \end{cases}$	$1^{(n-1)-x_1u_kc_k},$ $1^{(n-1)-\ u_kc_k},$	$(2x_0-1-2x_1)^{x_1u_kc_k},$ $(2x_0-1-2x_1)^{\ u_kc_k},$	$(2x_1)^{u_{k-1}c_k+1},$ $(2x_1)^{u_{k-1}c_k+1},$	k even. k odd.
(4c)	$\begin{cases} n=[x_1c_kd_k+x_1+1]; \\ n=[\ \ c_kd_k+x_1+1]; \end{cases}$	$1^{(n-1)-x_1u_kd_k},$ $1^{(n-1)-\ u_kd_k},$	$(2x_0-1-2x_1)^{x_1u_kd_k},$ $(2x_0-1-2x_1)^{\ u_kd_k},$	$(2x_1)^{u_{k+1}d_k+1},$ $(2x_1)^{u_{k+1}d_k+1},$	k even. k odd.
(5)	$\begin{cases} n=[e_{k+1}e_{k-1}+x_0(x_1-x_0+1)+1]; \\ n=[e_k^2+1]; \end{cases}$	$1^{(n-1)-u_{k+1}e_{k-1}-x_1+x_0},$ $1^{(n-1)-u_ke_k},$	$(2x_0-1)^{u_{k+2}e_{k-1}+x_1-x_0+1},$ $(2x_0-1)^{u_{k+1}e_k},$	$(2x_1-2x_0+1)^{u_ke_{k-1}+1},$ $(2x_1-2x_0+1)^{u_{k-1}e_k},$	k even. k odd.
(5′)	$\begin{cases} n=[f_{k-1}f_{k+1}+x_0(x_1-x_0+1)+1]; \\ n=[f_k^2+1]; \end{cases}$	$1^{(n-1)-u_{k-1}f_{k+1}-x_1+x_0},$ $1^{(n-1)-u_kf_k},$	$(2x_0-1)^{u_{k-2}f_{k+1}+x_1-x_0+1},$ $(2x_0-1)^{u_{k-1}f_k},$	$(2x_1-2x_0+1)^{u_kf_{k+1}+1},$ $(2x_1-2x_0+1)^{u_{k+1}f_k},$	k even. k odd.
(6)	$\begin{cases} n=[f_{k+1}e_{k-1}+(x_1-x_0)^2+1]; \\ n=[f_ke_k+x_1+1]; \end{cases}$	$1^{(n-1)-u_{k+1}e_{k-1}-x_1+x_0},$ $1^{(n-1)-u_ke_k},$	$(2x_0-1)^{u_ke_{k-1}},$ $(2x_0-1)^{u_{k-1}e_k+1},$	$(2x_1-2x_0+1)^{u_{k+2}e_{k-1}+x_1-x_0},$ $(2x_1-2x_0+1)^{u_{k+1}e_k+1},$	k even. k odd.
(6c)	$\begin{cases} n=[e_{k-1}f_{k+1}+(x_1-x_0)^2+1]; \\ n=[e_kf_k+x_1+1]; \end{cases}$	$1^{(n-1)-u_{k-1}f_{k+1}-x_1+x_0},$ $1^{(n-1)-u_kf_k},$	$(2x_0-1)^{u_kf_{k+1}},$ $(2x_0-1)^{u_{k+1}f_k+1},$	$(2x_1-2x_0+1)^{u_{k-2}f_{k+1}+x_1-x_0},$ $(2x_1-2x_0+1)^{u_{k-1}f_k+1},$	k even. k odd.

$a_i = x_0u_i + u_{i-2}$, all i.
$b_i = x_0u_i + u_{i+2}$, all i.

$c_i = (x_0-2)u_i - u_{i-2}$, all i.
$d_i = (x_0-2)u_i - u_{i+2}$, all i.

$e_i = x_0(u_{i+1}-u_{i-1})+(u_{i-1}+u_{i-2})$, i odd.
$f_i = x_0(u_{i-1}-u_{i+1})+(u_{i+1}+u_{i+2})$, i odd.

17. A necessary and sufficient condition that a characteristic be geometric is that there exist a set of T_2's with which it may be successively compounded so that the resultant is the linear characteristic. If the characteristic is geometric its conjugate may be found by compounding the linear characteristic (written so as to contain a basis point of multiplicity zero to correspond to each basis point of the given characteristic) with the same set of T_2's taken in the same order. Triadics 1*c*, 4*c*, and 6*c* may be derived in this way as the conjugates of triadics 1, 4, and 6, while all the others in the preceding table are self-conjugate.

Unfortunately the algebra involved in the investigations of these triadics is long and complicated, so that we shall merely sketch the outline of the procedure, using triadic 1 to illustrate it. Let the linear characteristic be written under, and in the same form as, triadic 1. Let P be the single point of the first group (see paragraph 2). Compound with T_2's having two basis points in the second group so that this group is reduced to a group of simple points. (One T_2 will have only one basis point in the second group and one general.) Reduce the multiplicity of the third group by compounding with T_2's having two basis points in it. If the multiplicity of the new third group is repeatedly reduced in the same way it is found that the formulas for all of the multiplicities change according to a certain simple law, so that the formulas for the multiplicities resulting from an arbitrary number of such reductions can be calculated directly. The number of such reductions needed is approximately k. When the triadic has been reduced in this way to the simplest possible form it is finally compounded with a set of T_2's which obliterates the group of simple points that has been carried along, and the result is the linear characteristic. In the meantime the same sets of T_2's have operated on the original linear characteristic to produce triadic 1*c*. Algebraic simplifications in this procedure are made with the aid of relations (B) and (C).

18. The conjugates of triadics 1, 4, and 6 may be formally obtained by substituting $-k$ for k in these triadics when they are written explicitly in terms of the u's. Triadics 2′, 3′, and 5′ were obtained by this same formal operation when applied to triadics 2, 3, and 5 respectively.

19. In paragraph 14 the case $k = 1$ was not considered. Triadics may be obtained from (5), paragraph 3, either by making one of the base numbers vanish or by making two of the multiplicities equal. Triadics 7, 8, 9, 10, and 11 result respectively from the conditions $2x_{01} - 1 - x_{12} = 0$, $x_{12} = 0$, $1 + 2x_{11} - x_{12} = 0$, $x_{01} - 1 = 1$, and $1 + x_{01}x_{11} - x_{12} = x_{01}$. Their conjugates are easily derived by Montesano's method. We write merely x_0 for x_{01}, x_1 for x_{11}, and x_2 for x_{12}.

(7)	$n = [x_1x_0 - x_0 + 2]$;	$1^{x_1x_0-2x_0+2}$,	$(2x_1 - 2x_0 + 2)^{x_0}$,		$(2x_0 - 1)^{x_0-1}$.
(7c)	$n = [x_1x_0 - x_0 + 2]$;	$1^{x_1x_0-x_1-x_0+1}$,	$(2x_1 - 2x_0 + 2)^{1}$,		$(2x_0 - 1)^{x_1}$.
(8)	$n = [x_1x_0 + x_0 + 1]$;	$1^{x_1x_0+1}$,	$(2x_1 + 1)^{x_0}$,		$(2x_0 - 1)^{1}$.
(8c)	$n = [x_1x_0 + x_0 + 1]$;	$1^{x_1x_0-x_1+x_0}$,	$(2x_1 + 1)^{1}$,		$(2x_0 - 1)^{x_1+1}$.
(9)	$n = [x_1x_0 - 2x_1 + x_0]$;	$1^{x_1x_0-2x_1}$,	$(2x_1 + 1)^{x_0-1}$,		$(2x_0 - 2x_1 - 2)^{1}$.
(9c)	$n = [x_1x_0 - 2x_1 + x_0]$;	$1^{x_1x_0-3x_1+x_0-1}$,	$(2x_1 + 1)^{x_1}$,		$(2x_0 - 2x_1 - 2)^{x_1+1}$.
(10)	$n = [2x_1 - x_2 + 3]$;	$1^{2x_1-x_2+1}$,	$(1 + 2x_1 - x_2)^{2}$,	x_2^{1},	$(3 - x_2)^{1}$.
(10c)	$n = [2x_1 - x_2 + 3]$;	$1^{x_1-x_2+2}$,	$(1 + 2x_1 - x_2)^{1}$,	$x_2^{x_1}$,	$(3 - x_2)^{x_1+1}$.
(11)	$n = [2x_0]$;	1^{x_0},	$(2x_1 + x_0 - x_1x_0)^{x_0}$,	$(1 + x_1x_0 - x_0)^{x_0-1}$,	$(3x_0 - x_1x_0 - 2)^{1}$.
(11c)	$n = [2x_0]$;	$1^{2x_0-x_1-1}$,	$(2x_1 + x_0 - x_1x_0)^{1}$,	$(1 + x_1x_0 - x_0)^{x_1}$,	$(3x_0 - x_1x_0 - 2)^{x_1+1}$.

CORNELL UNIVERSITY,
MARCH 25, 1931.

A DETERMINATION OF THE DOMAINS OF INTEGRITY OF THE COMPLETE RATIONAL MATRIC ALGEBRA OF ORDER 4.*

By E. J. FINAN.

1. *Introduction.* The set of all two-rowed square matrices with rational elements constitutes the complete matric algebra of order 4. We shall call a *domain of integrity* (or merely a *domain*) of this algebra any subset which (1) is of order 4, (2) contains the identity matrix, and (3) is closed under multiplication, the constants of multiplication being rational integers. Every such set has a basis † having the above three properties.

If E_1, E_2, E_3, and E_4 is a basis for such a domain, it is well known that E'_1, E'_2, E'_3, and E'_4 also constitute a basis where

$$E'_i = \sum_{j=1}^{4} t_{ij} E_j \qquad (i = 1, 2, 3, 4),$$

the t_{ij} being rational integers such that $| t_{ij} | = \pm 1$, and that every basis of the domain is so obtainable. We shall call this change of basis a transformation (1).

It is evident that if M is any non-singular two-rowed square matrix with rational elements the domain with basis E'_1, E'_2, E'_3, and E'_4 is isomorphic with the domain whose basis is

$$E'_i = ME_iM^{-1} \qquad (i = 1, 2, 3, 4).$$

If two domains are so related we shall call them *equivalent*. We shall call the process of obtaining an equivalent domain a transformation (2).

Since transformations (1) and (2) are both associative and commutative, we may consider any sequence of transformations as a transformation (1) followed by a transformation (2). We shall use this result later to show that two domains with the same discriminant may be non-equivalent.

Since the constants of multiplication remain unchanged, the discriminant is invariant under transformation (2). Also it is invariant under a transformation (1).‡

* Presented to the Society, April 19, 1930.

† Dickson, *Algebras and Their Arithmetics*, Chicago (1923). Page 161.

‡ MacDuffee, *Annals of Mathematics*, Vol. 2, No. 2: 2nd series.

There is an infinite number of non-equivalent domains, but it is shown in this paper that for a given value of the discriminant there is only a finite number. In § 2 we obtain two canonical forms for the basis of a domain. By applying to these canonical forms arbitrary transformations (1) and (2) all sets of matrices are obtained which constitute bases for domains of integrity. Finally the number of non-equivalent domains is calculated for small values of the discriminant and a canonical form is obtained for each.

This problem was first attacked by Du Pasquier,* who obtained after a more laborious reduction a reduced form which included all domains, unfortunately with much repetition. His attempt to classify domains by means of the greatest common denominator in his canonical form was not successful, since this function is not an invariant.

2. THEOREM 1. *Every domain of integrity is equivalent under transformations* (1) *and* (2) *to one of the domains.*

Case I

$$\left\|\begin{matrix}1 & 0\\ 0 & 1\end{matrix}\right\| \quad \left\|\begin{matrix}0 & 1\\ m & 0\end{matrix}\right\| \quad \left\|\begin{matrix}0 & 0\\ la & a\end{matrix}\right\| \quad \left\|\begin{matrix}0 & 0\\ ka & 0\end{matrix}\right\|$$

Case II

$$\left\|\begin{matrix}1 & 0\\ 0 & 1\end{matrix}\right\| \quad \left\|\begin{matrix}0 & 1\\ m & 1\end{matrix}\right\| \quad \left\|\begin{matrix}0 & 0\\ la & a\end{matrix}\right\| \quad \left\|\begin{matrix}0 & 0\\ ka & 0\end{matrix}\right\|$$

in which a, k, l, m *are rational integers,* $0 \leq l < k$, $0 \leq m < ka$ *and in Case I* $l^2 - m \equiv 0$, *mod.* k *while in Case II* $l^2 + l - m \equiv 0$, *mod.* k. *Conversely, every set of matrices of this form, satisfying these conditions, constitutes a basis of some domain.*

By a theorem due to Du Pasquier in the article mentioned above, we may assume that the basal matrices have integral elements.

Let this basis be E_1, E_2, E_3, and E_4 in which E_1 is the second order identity matrix. E_1 does not change throughout the reduction.

Evidently the operation of replacing a single E_j $(j = 2, 3, 4)$ by $E_j - xE_k$ $(k = 1, 2, 3, 4;\ j \neq k)$ is a transformation (1). We shall call this kind of transformation (1) a transformation (1-A).

By a transformation (1-A) with $j = 2$ and $k = 1$ we can put E_2 into the form

$$\left\|\begin{matrix}0 & b\\ c & d\end{matrix}\right\|.$$

* *Vierteljahrsschrift der Naturforshcenden Gesellschaft in Zurich* (1909), page 137.

A transformation (2) with

$$M = \begin{Vmatrix} 1 & 0 \\ 0 & b \end{Vmatrix}$$

then replaces E_2 by a matrix of the same form with $b = 1$. Another transformation (2) with

$$M = \begin{Vmatrix} 1 & 0 \\ -n & 1 \end{Vmatrix}$$

followed by a transformation (1-A) with $j = 2$, $k = 1$ and $x = n$ replaces E_2 by

$$\begin{Vmatrix} 0 & 1 \\ e & d - 2n \end{Vmatrix}.$$

By suitable choice of n we can make the lower right element in E_2 0 or 1 according as d is even or odd. We shall now divide the discussion into two cases.

Case I. d is even. E_2 is now in the form

$$\begin{Vmatrix} 0 & 1 \\ e & 0 \end{Vmatrix}$$

By a transformation (1-A) we can subtract multiples of E_1 and E_2 from E_3 and E_4 and give them the form

$$E_3 = \begin{Vmatrix} 0 & 0 \\ f & g \end{Vmatrix} \qquad E_4 = \begin{Vmatrix} 0 & 0 \\ p & q \end{Vmatrix}.$$

There exist relatively prime integers, λ and μ, such that $\lambda g + \mu q = 0$. Also there exist integers, ξ and η, such that $\mu\xi - \lambda\eta = 1$. The transformation (1) with matrix

$$\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \xi & \eta \\ 0 & 0 & \lambda & \mu \end{Vmatrix}$$

leaves E_1 and E_2 unchanged, replaces E_3 by a matrix of the same form and replaces E_4 by

$$\begin{Vmatrix} 0 & 0 \\ r & 0 \end{Vmatrix}.$$

If we now express the products E_4E_3 and E_4E_2 as linear combinations of E_1, E_2, E_3 and E_4 with integral coefficients, we get certain relations between

r, f, g and e. After some simplification these amount to $r = kg$, $f = lg$ and $l^2 \equiv e$, mod. k. Hence the basal matrices may be written

$$\left\|\begin{matrix}1 & 0\\ 0 & 1\end{matrix}\right\|, \quad \left\|\begin{matrix}0 & 1\\ m & 0\end{matrix}\right\|, \quad \left\|\begin{matrix}0 & 0\\ la & a\end{matrix}\right\|, \quad \left\|\begin{matrix}0 & 0\\ ka & 0\end{matrix}\right\|,$$

in which $l^2 \equiv m$, mod. k and $ka \neq 0$. The latter condition follows from the linear independence of the basal matrices.

We shall now show that it is possible by transformation (1) to make all of the elements positive or zero. If ka is negative, apply transformation (1) with T equal to the identity matrix with the exception that $t_{44} = -1$. If a is negative, use a similar transformation with $t_{33} = -1$. Also, with ka positive, we can make $0 \leqq m < ka$ and $0 \leqq l < k$ by repeatedly adding E_4 to E_2 and E_3 respectively. None of these transformations changes the conditions on the elements of the E's which are given above, for the 1 in E_2 is not changed, and none of the zeros is lost in any of the E's. Hence we have arrived at the canonical form given in the theorem.

Case II. d is odd. The procedure is similar to Case I. We shall not include it here. The conditions on the elements of the E's are given in the statement of the theorem.

To show that any four matrices satisfying the conditions given in the theorem constitute a basis for some domain, it is sufficient to construct the multiplication table of the matrices. This shows that the constants of multiplication are integers. The other conditions for a domain are evidently satisfied.

3. *Conditions for equivalence of two domains.* In our investigation we shall need the discriminant matrix of the domain. This is given by the formula *

$$D_{rs} = \sum_{ji} c_{rij} c_{sji}$$

in which the c's are the 64 constants of multiplication. Using the basis given in Theorem 1, the discriminant matrix for Case (1) is

$$\left\|\begin{matrix}4 & 0 & 2a & 0\\ 0 & 4m & 2la & 2ka\\ 2a & 2la & 2a^2 & 0\\ 0 & 2ka & 0 & 0\end{matrix}\right\|.$$

As mentioned in § 1 $|D| = -16k^2a^4$ is an invariant under transfor-

* MacDuffee, *loc. cit.*

mations (1) and (2). From Theorem 1 it is evident that there is only a finite number of non-equivalent domains for a given value of $|D|$. Hence using a constant ka^2, which is evidently the simplest function of the invariant, we can make a determination of the non-equivalent domains with a given discriminant. ($|D|$ is called the discriminant of the domain.) Although we have no formula which gives the number of non-equivalent domains for a given value of ka^2, we shall give 6 auxiliary theorems which are useful in determining this number. The rest of this paper will be devoted to the determination of the number of non-equivalent domains for values of ka^2 from 1 to 31 inclusive. In many cases the calculations are long; hence we are including only one typical calculation. The results are given in tabulated form at the end of the paper.

If two domains are equivalent * under transformations (1) and (2), it is necessary that there exist a matrix T of determinant ± 1 such that

$$T D \bar{T} = D'$$

where D and D' are the discriminant matrices of the two domains and $\bar{T} = T$ *transpose*. Also, since each domain contains the principal unit, it is still quite general if we demand that the first row of T be all zeros except the first element, which is unity.

The last equation evidently gives certain relations between the elements of T, D, and D'. There are 3 sets of such relations that we shall need; one when D and D' both come from domains of type (I); one when both D and D' come from domains of type (II); and one when D is of type (I) and D' is of type (II). However, we shall include only those equations which must hold when both domains are of type (I). There are 9 independent ones and we shall refer to them as A-1, A-2, etc. The theorems which follow from the other sets of equations will be stated without proof. In the following equations the unprimed letters are elements of the basal matrices whose discriminant matrix is D; the primed letters belong to the basal matrices whose discriminant matrix is D'. These A-conditions which must hold when two domains of type (I) are equivalent are

A-1 $2t_{21} + at_{23} = 0,$
A-2 $2t_{41} + at_{43} = 0,$
A-3 $2t_{31} + at_{33} = a',$
A-4 $t^2_{21} + mt^2_{22} + lat_{22}t_{23} + kat_{22}t_{24} = m',$

* MacDuffee, *loc. cit.*

A-5 $-t^2_{31} + mt^2_{32} + lat_{32}t_{33} + kat_{32}t_{34} - at_{33}t_{31} = 0,$

A-6 $t^2_{41} + mt^2_{42} + lat_{42}t_{43} + kat_{42}t_{44} = 0,$

A-7 $-at_{33}t_{21} + 2mt_{32}t_{22} + lat_{33}t_{22} + kat_{34}t_{22} + lat_{32}t_{23} + kat_{32}t_{24} = l'a',$

A-8 $-at_{41}t_{23} + 2mt_{42}t_{22} + lat_{43}t_{22} + kat_{44}t_{22} + lat_{42}t_{23} + kat_{42}t_{24} = k'a',$

A-9 $-at_{41}t_{33} + 2mt_{42}t_{32} + lat_{43}t_{32} + kat_{44}t_{32} + lat_{42}t_{33} + kat_{42}t_{34} = 0.$

4. *Auxiliary theorems.* The following theorems are useful in determining the number of non-equivalent domains with a given discriminant.

THEOREM 2. *If* $2l \equiv 0$, *mod.* a, *the domain is equivalent to some domain in canonical form in which* $l = 0$.

To prove this it is sufficient to give the transformation which makes l zero without changing the canonical form given in Theorem 1. The following transformation works for both cases.

$$T = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & \frac{-2l}{a} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix},$$

and

$$M = \begin{Vmatrix} -1 & 0 \\ -l & -1 \end{Vmatrix}.$$

This transformation leaves k and a invariant. If m' is such that the domain is not in canonical form, this can be corrected by adding or subtracting ka a sufficient number of times.

THEOREM 3. *Any domain of type* (*II*) *with* a *odd is equivalent to a domain of type* (*I*).

The following transformation will put the domain of type (II) into one of type (I).

$$T = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ \frac{1+a}{-2} & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix},$$

and

$$M = \begin{Vmatrix} -1 & 0 \\ \frac{1+a}{2} & -1 \end{Vmatrix}.$$

THEOREM 4. *If ka^2 contains no square factor other than unity, all domains are equivalent.*

By Theorem 3 we need consider only domains of type (I), since a is evidently 1. By Theorem 2 we may take l as zero. Hence

$$m \equiv 0, \text{ mod. } k.$$

But by subtracting or adding k we can evidently make m zero. Hence m and l are both zero and a is 1. Hence there is only one such domain for such a value of ka.[2]

THEOREM 5. *Two domains of type I are non-equivalent if a is even in one and odd in the other.*

If a' is odd A-3 is impossible unless a is even. This proves the theorem.

THEOREM 6. *A given domain of type (II) is not equivalent to a domain of type (I) with a even.*

THEOREM 7. *Two domains of type (II) in which the a's are even are not equivalent unless the m's are both even or both odd.*

5. *Non-equivalent domains when $ka^2 = 18$.* We shall now make a determination of the number of non-equivalent domains for which $ka^2 = 18$. This is a typical case. Since 1 and 9 are the only square factors in 18 it is, by Theorem 3, sufficient to consider only domains of type (I). The following seven domains are the only ones satisfying the conditions given in Theorem 1.

	k	a	l	m
D	18	1	0	0
D	2	3	0	0
D	2	3	0	2
D	2	3	0	4
D	2	3	1	1
D	2	3	1	3
D	2	3	1	5

The domains D_1 and D_2 are non-equivalent, for if we apply a transformation (1) to D_2, we get

$$\text{(A-5)} \qquad -t^2_{31} + 6t_{32}t_{34} - 3t_{33}t_{31} = 0,$$
$$\text{(A-1)} \qquad 2t_{31} + 3t_{33} = 1.$$

The first of the above equations demands that $t_{31} \equiv 0$, mod. 3, and this is impossible in the second one.

Also D_1 and D_3 are non-equivalent, for if we apply transformation (1) to D_1, A-4 becomes

$$t^2_{21} + 18t_{22}t_{24} = 2.$$

But $t^2 \equiv 2$, mod. 3 has no roots. Hence D_1 and D_3 are non-equivalent.

Now D_2 and D_3 are non-equivalent, for if we apply a transformation (1) to D_2 we have from A-1 and A-4 respectively:

$$\begin{aligned} 2t_{21} + 3t_{23} &= 0, \\ t^2_{21} + 6t_{22}t_{24} &= 2, \end{aligned}$$

which have no common solutions.

The transformations

$$T = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 6 & -3 & -4 & 4 \\ 2 & -1 & -1 & 1 \\ -6 & 6 & 4 & -5 \end{Vmatrix},$$

and

$$M = \begin{Vmatrix} -1 & 1 \\ 6 & -3 \end{Vmatrix}$$

put D_4 into D_1. Also the transformations

$$T = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix},$$

and

$$M = \begin{Vmatrix} -1 & 0 \\ 3 & -1 \end{Vmatrix},$$

put D_2, D_3, and D_4 into domains which are evidently equivalent to D_6, D_7, and D_5 respectively.

This concludes the determination. There are evidently just three non-equivalent domains. These are D_1, D_2, and D_3 which are listed above.

6. *Canonical form for all non-equivalent domains with* $ka^2 \leq 31$. We shall exhibit below a table of the non-equivalent domains for values of ka^2 from 1 to 31. Since by Theorem 4 all domains are equivalent when ka^2 contains no square factor other than one, we shall tabulate only the domains for the remaining values of the invariant.

In the table the letters have the same meaning as in the canonical form of Theorem 1.

ka^2	k	a	l	m	Type
1	1	1	0	0	I
4	4	1	0	0	I
4	1	2	0	0	I
4	1	2	0	1	II
8	8	1	0	0	I
8	2	2	0	0	I
8	2	2	0	2	I
9	9	1	0	0	I
9	1	3	0	0	I
9	1	3	0	2	I
12	12	1	0	0	I
12	3	2	0	3	I
12	3	2	0	3	II
16	16	1	0	0	I
16	4	2	0	0	I
16	4	2	0	4	I
16	1	4	0	0	I
16	1	4	0	2	I
16	1	4	0	3	I
16	1	4	0	1	II
18	18	1	0	0	I
18	2	3	0	0	I
18	2	3	0	2	I
20	20	1	0	0	I
20	5	2	0	0	I
20	5	2	0	5	II
24	24	1	0	0	I
24	6	2	0	0	I
24	6	2	0	6	I
25	25	1	0	0	I
25	1	5	0	0	I
25	1	5	0	2	I
27	27	1	0	0	I
27	3	3	0	0	I
27	3	3	0	3	I
27	3	3	0	6	I
27	3	3	1	1	I
28	28	1	0	0	I
28	7	2	0	0	I
28	7	2	0	7	II

Summary of above results.

Value of ka^2	1	4	8	9	12	16	18	20	24	25	27	28
Number of Domains . . .	1	3	3	3	3	7	3	3	3	3	5	3

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PENCILS OF HYPERSURFACES.*

By TEMPLE RICE HOLLCROFT.

1. *Introduction.* A general pencil f of hypersurfaces in r-space S_r is defined by the equation

$$f \equiv \lambda_1 u_1 + \lambda_2 u_2 = 0, \tag{1}$$

in which u_1 and u_2 are given, non-singular hypersurfaces of S_r and $\lambda \equiv \lambda_1/\lambda_2$ is the parameter.

Pencils of plane curves were first studied systematically by Cremona.† Certain properties of pencils of surfaces of S_3 have been found by various mathematicians, but in most cases only as special properties of a k-parameter linear system for $k = 1$. Doehlemann ‡ devotes a short section of his article to properties peculiar to pencils.

The purpose of this paper is to determine the characteristics of a general pencil of hypersurfaces in r dimensions.

2. *The basis manifold.* The basis manifold M of the pencil f is of dimension $r-2$ and order n^2. It is the complete intersection of u_1 and u_2 or of any two, three, $\cdots$, infinity hypersurfaces of f.

The i-th class m_i of a manifold M_α of dimension α in S_r, $\alpha \leq r-1$, is the class of the section of M_α made by an arbitrary $S_{r-\alpha+i}$; that is, the number of $S_{r-\alpha+i-1}$ through an arbitrary $S_{r-\alpha+i-2}$ of $S_{r-\alpha+i}$ that are tangent to the section of M_α made by $S_{r-\alpha+i}$. For a general hypersurface of order m_0, $m_i = m_0(m_0-1)^i$.

The basis manifold M of dimension $r-2$ is the complete intersection of two hypersurfaces in S_r. The i-th class of M is the i-th class of the manifold (hypersurface of S_{i+1}) that is the projection on S_{i+1} from an arbitrary point of S_{i+2} of the section of M made by S_{i+2}. The basis manifold M has, therefore, $r-2$ distinct classes, m_i, $i = 1, 2, \cdots, r-2$, which are respectively the classes of the sections of M made by an arbitrary S_{i+2}.

Under the above definition, the order of a manifold may be considered as

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* Presented to the American Mathematical Society, September 9, 1930.

† L. Cremona, "Einleitung in eine geometrische Theorie der Ebenen Curven," *Greifswald*, 1865, pp. 116-130.

‡ K. Doehlemann, "Lineare Systeme von Curven und Flächen," *Mathematische Annalen*, Vol. 41 (1893), pp. 560-563.

the class m_0; that is for M, the class of the section of M by an arbitrary plane which is n^2.

The class m_1 of M is the class of a 3-space section of M. This is the rank of a space curve of order n^2 (or class of the plane curve projection of this space curve from a point of S_3) which is the complete intersection of two surfaces of S_3 of order n. Then $m_1 = n^2(n^2-1) - n^2(n-1)^2 = 2n^2(n-1)$.

The class m_2 of M is the class of a 4-space surface (or class of the 3-space surface into which this surface projects from a point of S_4) which is the complete intersection of two hypersurfaces of order n of S_4. Since such a surface has no improper double points, its class

$$m_2 = 3t + \tfrac{1}{2}m_0\,[3m_1 - (m_0-1)(m_0+2)],^*$$

wherein the order $m_0 = n^2$, the class of a 3-space section $m_1 = 2n^2(n-1)$, the number of apparent triple points $t = n^2(n-1)^2(n-2)^2/6$. Hence $m_2 = 3n^2(n-1)^2$.

Comparing the above three values of m_i for $i = 0, 1, 2$, it is evident that the values of the classes m_i of M are

$$m_i = (i+1)n^2(n-1)^i, \qquad (i = 1, 2, 3, \cdots, r-2).$$

Using the formula for the equivalence † of a manifold of given dimension, order and classes which occurs simply on r hypersurfaces of given order in S_r, the equivalence of the basis manifold M on r hypersurfaces of order N in S_r is found to be

$$E_M = n^2\,[{}_rC_2N^{r-2} - 2{}_rC_3nN^{r-3} + 3{}_rC_4n^2N^{r-4} - \cdots + (-1)^{r-1}(r-2){}_rC_1n^{r-3}N + (-1)^r(r-1)n^{r-2}].‡$$

For $r = 2$ and all values of N, E_M has the value n^2, that is, the equivalence of n^2 points common to any two curves is n^2. Also, E_M has the value n^r for $n = N$, that is, in S_r, M is the complete intersection of r hypersurfaces of order n.

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* F. Severi, "Intorno ai punti impropri di una superficie generale dello spazio a quattro dimensioni e suoi punti tripli apparenti," *Rendiconti del Circolo Matematico di Palermo*, Vol. 15 (1901), pp. 34-36.

† F. Severi, "Sulle intersezioni delle varieta algebriche e sopra i loro caratteri e singolarità proiettive," *Memorie delle Reale Accademia delle Scienze di Torino*, Ser. 2, Vol. 52 (1903), p. 115.

‡ The expression resulting directly from substitution in the formula is in powers of $N-1$ with increasingly long coefficients. On expanding the binomials $(N-1)^i$ and combining like terms, the above comparatively simple form is obtained.

3. *Hypersurfaces and systems of hypersurfaces associated with the pencil.* The k-polar hypersurfaces of a pencil of hypersurfaces form systems, linear in the parameter λ and of degree k in the r independent coördinates of a general point of S_r.

The Hessian of a pencil of hypersurfaces of order n is a one-parameter family of hypersurfaces of order $(r+1)(n-2)$ in whose equation the parameter λ occurs to the degree $r+1$. The Hessian does not contain M.

One point P given on the Hessian determines the Hessian by means of an equation of degree $r+1$ in λ. Then a point given on the Hessian of a pencil of hypersurfaces determines $r+1$ hypersurfaces of the pencil and the $r+1$ Hessians associated with them. From the definitions of the Hessian as a locus, we have, therefore: A given point P considered either as a contact of two first polars, as a node * on a first polar, or as a pole whose $(n-2)$-polar is a quadric hypercone (all polars with respect to the pencil of hypersurfaces) determines $r+1$ hypersurfaces of the pencil whose $r+1$ respective Hessians all pass through P.

Eliminating λ from the equations of the pencil and the Hessian, the equation of a fixed hypersurface J of order $2(r+1)(n-1)$ is obtained. This hypersurface contains M $(r+1)$-fold and is the locus of the parabolic manifolds of hypersurfaces of the pencil.

The envelope of the Hessians of a pencil of hypersurfaces of order n is a unique hypersurface of order $(2r+1)(n-2)$ which does not contain M.

The above results for the first Hessian can easily be extended to any k-Hessian, $1 \leq k \leq n-2$,† which includes the Steinerians of the pencil.

The k-polar hypersurfaces of a fixed point P with respect to the hypersurfaces of the pencil themselves form a pencil of hypersurfaces of order $n-k$ with a basis manifold of order $(n-k)^2$ and dimension $r-2$. These pencils are all projective to the original pencil and to each other for the same or different positions of P, since all are linear one-parameter systems with the same parameter λ. A multiple infinity of fixed hypersurfaces associated with the pencil can be determined as loci of the intersections of corresponding hypersurfaces of any two of these projective pencils.

The most important of these fixed hypersurfaces is the locus of intersections of corresponding hypersurfaces of the original pencil and the first polar pencil of a given point P. This hypersurface ϕ_P is of order $2n-1$. It is the locus of contacts of all tangent hyperplanes from P to hypersurfaces

* The term "node" of a hypersurface is used to mean a multiple point of order two with an entirely general tangent quadric hypercone.

† T. R. Hollcroft, "The Generalized Hessian," *Quarterly Journal of Mathematics*, Vol. 50 (1927), pp. 362-372.

of the pencil. It contains the basis manifold M and passes through the nodes of hypersurfaces of the pencil.

For all points of S_r, the hypersurfaces ϕ_P form an ∞^r system of surfaces of order $2n-1$. This system has M as a basis manifold and the nodes of the pencil as simple basis points. The Jacobian of this system is of order $2(r+1)(n-1)$, contains M $(r+1)$-fold and has nodes at the nodes of the pencil.

This Jacobian hypersurface J is identical with the hypersurface J found above by eliminating λ from the equations of the pencil and the Hessian. This is easily seen by writing out the equation of J according to both definitions. This equation is a determinant, equated to zero, of order $r+1$, whose principal diagonal contains the elements

$$u_2\partial^2 u_1/\partial x_i^2 - u_1\partial^2 u_2/\partial x_i^2, \qquad (i=1, 2, \cdots, r+1).$$

4. *Nodes of the pencil.* Algebraically, the determination of the number H of hypersurfaces of the pencil that have nodes consists merely in finding the order of the discriminant of a quantic of order n in r independent variables. This discriminant is of order $(r+1)(n-1)^r$. Hence

$$H=(r+1)(n-1)^r.$$

This result may also be obtained synthetically. Consider r hypersurfaces ϕ_{P_i}, defined in the preceding section, determined respectively by r distinct arbitrary points P_i of S_r.* Each of these r hypersurfaces passes through M, the H nodes of the pencil and the $r(n-1)^{r-1}$ contacts of hypersurfaces of the pencil with the S_{r-1} determined by the r points P_i (see section 5). Substituting $N=2n-1$ in the formula for E_M derived in section 2, there results, on expanding the binomials and simplifying,

$$E_M{}^{\phi} = n^2[{}_rC_2(n-1)^{r-2} + {}_rC_3 n(n-1)^{r-3} + \cdots + rn^{r-3}(n-1) + n^{r-2}].$$

Hence

$$H=(2n-1)^r - r(n-1)^{r-1} - E_M{}^{\phi} = (r+1)(n-1)^r.$$

Another method used by Cremona (*loc. cit.*, pp. 122-123) can be generalized. Consider the first polar pencils p_i respectively of $r+1$ distinct points P_i and the r hypersurfaces of order $2(n-1)$ generated by the r pairs of projective pencils p_1p_2, $p_1p_3, \cdots, p_1p_{r+1}$. These r hypersurfaces all pass through the nodes of the pencil and the basis manifold of p_1. This basis manifold is of order $(n-1)^2$ and its equivalence E' on r hypersurfaces of

* This method applied to pencils of plane curves was used by Guccia, *Rendiconti del Circolo Matematico di Palermo*, Vol. 9 (1895), p. 15.

order $2(n-1)$ is found by replacing N by $2(n-1)$ and n by $n-1$ in the formula for E_M. This gives

$$E' = (n-1)^r [{}_rC_2 2^{r-2} - {}_rC_3 2^{r-3} + \cdots + (-1)^r (r-1)] \equiv (n-1)^r (2^r - r - 1).$$

Hence

$$H = [2(n-1)]^r - E' = (r+1)(n-1)^r.$$

5. *The number of hypersurfaces of a pencil tangent to a given manifold.* We shall first find the number of hypersurfaces of a pencil tangent to a given non-singular hypersurface u of order m.

The solution of this problem is equivalent to finding the order of the tact-invariant T of u and a general hypersurface of the pencil. Since all coefficients of both u and the pencil are constants except λ, T is a function of λ. The roots of $T = 0$ are values of λ defining hypersurfaces of the pencil tangent to u.

To find the order of the tact-invariant T, we shall first consider the net of hypersurfaces in S_r,

$$\lambda u + \lambda_1 u_1 + \lambda_2 u_2 = 0,$$

in which $u = 0$, $u_1 = 0$, $u_2 = 0$ are hypersurfaces of orders m, n, p respectively. The locus of contacts of the hypersurfaces of the net is the Jacobian J, a hypercurve whose order will be found. Let $p = n$. Consider the two hypersurfaces $u = 0$ and $f \equiv \lambda_1 u_1 + \lambda_2 u_2 = 0$ of orders m and n respectively belonging to the net. The contacts of these two hypersurfaces will lie on J and will, therefore, lie at the intersections of u and J. The number of intersections of u and J will, therefore, be the order of the tact-invariant of u and f in the coefficients of f.

The equations of J are most easily derived from the following definition of J: The Jacobian of three hypersurfaces of S_r $u = 0$, $u_1 = 0$, $u_2 = 0$ of orders m, n, p respectively is the locus of points whose polar hyperplanes with respect to the three hypersurfaces have an S_{r-2} in common.

From this definition, the equations of J are given by the matrix

$$\left\| \begin{matrix} u' & u'' & u''' & \cdots & u^{(r+1)} \\ u'_1 & u''_1 & u'''_1 & \cdots & u_1^{(r+1)} \\ u'_2 & u''_2 & u'''_2 & \cdots & u_2^{(r+1)} \end{matrix} \right\|$$

in which the superscript i denotes the partial derivative with respect to x_i. The equations of J consist of $r-1$ independent determinants of this matrix, each equated to zero. Each of these $r-1$ equations represents a hypersurface of order $m+n+p-3$. The hypercurve J is the portion of the intersection of a set of $r-1$ independent hypersurfaces that is common to all the hypersurfaces belonging to the matrix.

Salmon * has derived a method for obtaining the order of the system common to all the determinants contained in any given matrix. Using his notation, we may write the above matrix, which contains 3 rows and $r-1$ columns, in the form

$$\left\| \begin{matrix} \alpha & \alpha & \alpha & \cdots & \alpha \\ \beta & \beta & \beta & \cdots & \beta \\ \gamma & \gamma & \gamma & \cdots & \gamma \end{matrix} \right\| .$$

The order of the system common to all the determinants of this matrix is

$$\sum_{0}^{r-1} \alpha^i \beta^j \gamma^k, \qquad (i+j+k=r-1);$$

that is, the sum of the $r(r+1)/2$ products $\alpha^i\beta^j\gamma^k$ in which i, j, k each ranges from 0 to $r-1$ subject to the restriction $i+j+k=r-1$.

Since $\alpha = m-1$, $\beta = n-1$, $\gamma = p-1$, the order of J is

$$\begin{aligned}(m-1)^{r-1} &+ (m-1)^{r-2}[(n-1)+(p-1)] \\ &+ (m-1)^{r-3}[(n-1)^2+(n-1)(p-1)+(p-1)^2] \\ &+ \cdots + (m-1)[(n-1)^{r-2}+(n-1)^{r-3}(p-1)+\cdots+(p-1)^{r-2}] \\ &+ (n-1)^{r-1}+(n-1)^{r-2}(p-1)+\cdots+(p-1)^{r-1}.\dagger\end{aligned}$$

For $p=n$, the order of J becomes

$$\begin{aligned}(m-1)^{r-1} &+ 2(m-1)^{r-2}(n-1)+3(m-1)^{r-3}(n-1)^2 \\ &+ \cdots + (r-1)(m-1)(n-1)^{r-2}+r(n-1)^{r-1}.\end{aligned}$$

The number of points common to u and J which is the order of the tact-invariant T of f and u in the coefficients of f is, therefore,

$$\begin{aligned}m[&(m-1)^{r-1}+2(m-1)^{r-2}(n-1) \\ &+ \cdots + (r-1)(m-1)(n-1)^{r-2}+r(n-1)^{r-1}].\end{aligned}$$

This is, moreover, the number of hypersurfaces of the pencil which are tangent to a given non-singular hypersurface of order m.

The above result is easily generalized to obtain the number of hypersurfaces of the pencil that are tangent to a given manifold M_α of order m and dimension $\alpha \leq r-1$.

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* G. Salmon, *Lessons Introductory to the Modern Higher Algebra*, 4th edition, pp. 286-290.

† Cf. T. R. Hollcroft, "Nets of Manifolds in i Dimensions," *Annali di Matematica*, Ser. 4, Vol. 5 (1927-28), pp. 261-267. The work of section 2 in finding the order of J and the tact-invariant is a special case of the above for $m=n=p$.

In $S_{\alpha+1}$, given a hypersurface M_α of order m. The contacts of members of the pencil with M_α will all lie in $S_{\alpha+1}$, that is, they will be the contacts of M_α and a general pencil of manifolds of order n and dimension α, the section of the pencil of S_r by $S_{\alpha+1}$. This is the same as the original problem for $r = \alpha + 1$. Then the number of hypersurfaces of a pencil of order n in S_r that are tangent to a given non-singular manifold of order m, dimension $\alpha \leq r-1$, which is contained in an $S_{\alpha+1}$, is

$$(2) \quad N_\alpha = m\,[(m-1)^\alpha + 2(m-1)^{\alpha-1}(n-1) + \cdots + (m-1)(n-1)^{\alpha-1} + (\alpha+1)(n-1)^\alpha].$$

For $m = 1$, formula (2) becomes $(\alpha+1)(n-1)^\alpha$, the number of hypersurfaces of the pencil tangent to a linear manifold S_α.

Formula (2) may be interpreted as follows: The pencil of hypersurfaces of order n of S_r intersects M_α in a pencil of manifolds of order mn and dimension $\alpha - 1$ lying in M_α. The number of manifolds of this pencil in M_α that have a node is N_α. This is evident because a node of a manifold in M_α occurs at a contact of M_α with a hypersurface of the pencil.

A hyperspace curve C of order m_0 intersects the pencil of hypersurfaces in a single infinity of groups of $m_0 n$ points, each group of which is such that any one point determines the remaining $m_0 n - 1$. If C is of genus p, this involution on C contains $2(m_0 n + p - 1)$ coincidences.* There are, then, $N'_1 = 2(m_0 n + p - 1)$ hypersurfaces of the pencil that are tangent to a curve of order m_0, genus p and belonging to any space.

This formula shows the effect only of nodes, apparent or actual, since the additional property of a cusp does not appear in a formula involving only the order and genus. From either the above formula or the succeeding one, N'_1 is reduced two by each additional node or apparent double point.

Let m_1 be the rank of C. Then $p = m_1 - 2(m_0 - 1)$ and the above formula becomes

$$(3_1) \qquad N'_1 \equiv m_1 + 2m_0(n-1)$$

where N'_1 is the number of hypersurfaces of the pencil tangent to a hyperspace curve C of order m_0 and rank m_1. It is now seen that each cusp reduces N'_1 by three. Also a general s-fold point of C reduces N'_1 by $s(s-1)$, since this reduction occurs in the rank.

In $S_{\alpha+1}$, consider a general manifold M_α of dimension α, order m_0 and classes $m_i = m_0(m_0-1)^i$, $(i = 1, 2, 3, \cdots, \alpha)$, such that $\alpha \leq r-1$. Sub-

* Pascal, *Repertorium der höheren Mathematik*, Vol. II$_1$ (1902), p. 153, line 3, $Q = m_0 n$, $q = 1$. The theorem cited deals with curves of S_3. By projection, it can be shown to hold for curves of any space.

stituting m_i for the respective products $m_0(m_0-1)^i$ in formula (2), there results: The number of hypersurfaces of a pencil of order n in S_r tangent to a manifold M_α of dimension α, order m_0 and classes $m_1, m_2, m_3, \cdots, m_\alpha$, is

$$(3) \quad N'_\alpha \equiv m_\alpha + 2m_{\alpha-1}(n-1) + 3m_{\alpha-2}(n-1)^2 + \cdots + \alpha m_1(n-1)^{\alpha-1} + (\alpha+1)m_0(n-1)^\alpha.$$

This is the general formula of which formula (3_1) above for the hyperspace curve C is a special case for $\alpha = 1$. Formula (3_1) can be derived from (2) by setting $\alpha = 1$ and $m_1 = m_0(m_0-1)$, the value of m_1 for a general plane curve. This is the same process by which (3) was derived from (2) for any α. Formula (3_1), however, was derived independently for a hyperspace curve of given order and class and therefore holds for a curve of any space.

Because of this and other evidences and analogies (e. g., to equivalence formulas), it would appear that formula (3) holds for manifolds of any order m_0, classes $m_i \leq m_0(m_0-1)^i$, and dimension α, defined by intersections of $r-\alpha$ linear or non-linear hypersurfaces of S_r, but this has not been proved.

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CORRECTIONS.

J. L. DORROH, *Some metric properties of descriptive planes.*

p. 403, l. 5 change "segment H_{jk}" to "segment of H_{jk}".

p. 407, l. 13 change "$|f(Z,V)-f(Z,x|$" to "$|f(Z,V)-f(Z,x)|$".

p. 407, l. 29 change "Y_e'" to "Y_e^*".

p. 415, l. 12 change "L" to "$\bar{L}$".

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